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Equilibria and Centrality in Link Formation Games

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ABSTRACT

We study non-cooperative link formation games in which players have to decide how much to invest in relationships with other players. The relationship between equilibrium strategies and network centrality measures are investigated in games where there is a common valuation of players as friends. If the utility from relationships with other players is bilinear, then indegree, eigenvector centrality, and the Katz-Bonacich centrality measure put the players in opposite order than the common valuation. If the utility from relationships is strictly concave, then these measures order the players in the same way as the common valuation.

JEL Classification: C72, D43

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1. Introduction

In social network analysis, centrality measures or indices attempt to measure the importance of an actor in a given network. The best known measures are the *indegree*, the *eigenvector centrality measure* (Bonacich 1972), the *Katz-Bonacich measure* (Katz 1953, Bonacich 1987, and Bonacich and Lloyd 2001), and of course the *PageRank* method (Brin and Page 1998).

The indegree of a node or an actor is the sum of intensities of links from the immediate neighbors of that actor. The Katz-Bonacich measure of an actor can be viewed as a discounted sum of intensities of links from neighbors, from neighbors of neighbors, and so on. The eigenvector centrality of an actor is a weighted sum of intensities of links from the immediate neighbors of that actor, where the weights are given by the eigenvector centralities themselves.

Kitti (2012) has given an axiomatic characterization of the eigenvector centrality measure. Altman and Tennenholz (2005) have given an axiomatic characterization of the PageRank method. The PageRank method is close to the Katz-Bonacich measure or the eigenvector centrality measure, depending of values of parameters chosen for the PageRank method (see Section 2). I am not aware of axiomatic characterizations of the Katz-Bonacich measure.

It is unclear who has first proposed the indegree as a centrality measure for directed networks, but see Freeman (1979) for a review of degree based measures for *undirected* networks. It seems that eigenvector centrality measure has been discussed already by Seeley (1949), Wei (1952), and Kendall (1955) (see *e.g.* Kitti 2012, Boldi and Vigna 2013). The Katz-Bonacich measure appears first time in Katz (1953).

Ballester *et.al* (2006) analyze a noncooperative game played in a *given* network of agents. Each agent chooses a single real number that describes his activity level in the whole network. Agent's utility is a quadratic function depending on the activity levels of all agents. Ballester *et.al* (2006) show that the equilibrium is proportional to the Katz-Bonacich measure (see also Jackson and Zenou 2014 for a review of related models).

We will analyze the relationship between equilibria of link formation games and centrality measures. Link formation games are models that try to describe the way networks are formed, and equilibria of these games give a prediction about how stable networks might look like. In the seminal paper Jackson and Wolinsky (1996) link formation strategy is a discrete $0-1$ variable. Bloch and Dutta 2009 analyze games in which link strengths are continuous variables.

In our model an agent has one unit of some resource like time or effort that he may use to form links with other players. The utility agent i gets from agent j is a Cobb-Douglas type of function of the investments of these players. This function is weighted by a positive parameter p_j that does not depend on i. Hence players have a common ranking over other players as friends. Total utility of a player is the sum of utilities from all relationships minus the opportunity cost of privacy.

We solve explicitly equilibrium strategies that produce a complete network and the centrality measures corresponding to these equilibrium networks. If utility functions are bilinear, then the indegree measure, the eigenvector centrality measure, and the Katz-Bonacich measure put the players in opposite order than players' original ranking given by the coefficients p_j. If utility functions are strictly concave, then these centrality measures order the players the same way as players' original ranking.

The model and notation is introduced in Section 2. The results are given in Section 3.

2. The Model

Given a finite set N, and a function $L : N \times N \longrightarrow \mathbb{R}^+$, the pair $W = (N, L)$ is called a (directed, weighted) network on N or simply a network. The number $L_{ij} \equiv L_(i, j)$ is the weight, strength, or intensity of the *link* from i to j. There is no link from i to j, if $L_{ij} = 0$, and we will assume $L_{ii} = 0$ for all $i \in N$. Network is directed because $L_{ij} = L_{ji}$ need not hold. A network $W' = (N', L')$ is a *subnetwork* of $W = (N, L)$, if

$N' \subset N$ and L' is the restriction $L_{|A}$ of L into N'. A network $W = (N, L)$ is complete if $L_{ij} > 0$ for all distinct nodes $i, j \in N$.

Given a network $W = (N, L)$ and $i, j \in N$, there exists a *path* P_{ij} from i to j, if there exists nodes $i_0, \ldots, i_K$ such that 1) $i_0 = i, i_K = j$; 2) $L_{i_k i_{k+1}} > 0$ for all $k = 0, \ldots, K-1$; 3) all nodes are distinct except possibly i_0 and i_K. A subset $A \subset N$ is *connected*, if for any $i, j \in A$, there is a path P_{ij} or P_{ji} that lies entirely in A. If paths P_{ij} and P_{ji} lie in A for all $i, j \in A$, then A is *strongly connected*. A network $W = (N, L)$ is connected if N is connected. We call a strongly connected network $W = (N, L)$ a *complete* network.

A subset $A \subset N$ is a *component* of a network $W = (N, L)$, if 1) A is connected; 2) there are no links between A and $A^c \equiv N \setminus A$. So a component A is a maximal strongly connected subset of N. A *clique* is a subset $A \subset N$ such that $L_{ij} > 0$ for all distinct nodes $i, j \in A$. If a component is a clique (*i.e.* the component is strongly connected), we may call it a complete component. If A is a clique then the subnetwork $W^A = (A, L_{|A})$ is a complete network.

The *indegree* d_i of node i is the number $\sum_{j \neq i} L_{ji}$, *i.e*, the sum of weight of links from nodes $j \neq i$ to i. We may interpret L as a matrix, and then d_i is the i'th column sum.

The *eigenvector centrality measure* is the left eigenvector q associated with the greatest eigenvalue λ of L. So q satisfies the equation $qL = \lambda q$. By the Perron-Frobenius theorem, if for each distinct i and j there exists a path P_{ij} (*i.e.* L is irreducible), then L has a greatest eigenvalue $\lambda > 0$. In case L is a stochastic matrix (*i.e.* all row sums equal 1), $\lambda = 1$ and $qL = q$, so q is a fixed point of L. In this paper we deal mostly with irreducible stochastic matrices.

The *Katz-Bonacich measure* b of a stochastic matrix L satisfies the equation $b(I - \alpha L) = \mathbf{1}$, where $\alpha \in (0, 1)$ is a given constant, I is the identity matrix, and $\mathbf{1} = (1, \ldots, 1)$. If L is irreducible, then b is strictly positive. This holds because the inverse matrix of $(I - \alpha L)$ is $\sum_{k=0}^{\infty} (\alpha L)^k$, and irreducibility means for each distinct i and j there is k such that the cell $(L^k)_{ij}$ of the matrix L^k is strictly positive.

The *PageRank measure* g^{pr} of a stochastic matrix L satisfies the equation $g^{pr} = \alpha q^{pr} L + (1-\alpha)w$, where $\alpha \in [0,1]$, and w is a strictly positive "preference vector" (Boldi and Vigna 2013). If $\alpha = 1$, then PageRank and the eigenvector centrality are the same. If $w = \mathbf{1}$ (like in the 1998 article of Brin and Page), then the PageRank measure is proportional to the Katz-Bonacich measure: $g^{pr} = (1-\alpha)b$. This follows because $g^{pr}(I - \alpha L) = (1-\alpha)w$.

A *normal form game* $G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big)$ specifies a player set N, a set of pure strategies S_i and a utility function $u_i : S \longrightarrow \mathbb{R}$ for each player $i \in N$, where $S = \Pi_i S_i$, the product of strategy sets, is the set of *strategy profiles.*

Give $s \in S$, we may denote $s = (s_i, s_{-i})$ when we want to emphasize that i chooses s_i. A pure strategy Nash equilibrium is a strategy profile $s \in S$ such that

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall i \in N, \forall s_i' \in S_i. \tag{1}$$

We study *link formation games* of the following type. The set of pure strategies of player $i \in N$ is

$$S_i = \Big\{ s_i \in R_+^N \mid \sum_j s_{ij} = 1 \Big\}.$$

An interpretation is that each player i has one unit of time or effort to be shared with other player j including i himself. The utility function of player i is

$$u_i(s) = \sum_{j \neq i} p_j s_{ij}^{\alpha} s_{ji}^{\beta} + c s_{ii}, \forall s \in S, \tag{2}$$

where $\alpha, \beta, c, p_j > 0$ for all j.

There is a common ordering of players such that player j is considered more valuable than i, if $p_j > p_i$. The cost parameters c reflects the opportunity cost of privacy. These games are special cases of so called semi-symmetric link formation games studied by Salonen (2014). We give a detailed analysis of two cases: 1) *bilinear games* for which $\alpha = \beta = 1$; 2) *concave games* for which $\alpha + \beta < 1$.

3. Results

We give first necessary and sufficient conditions for bilinear games to have an equilibrium with a complete network. We concentrate mostly on these kind of equilibria, since we want to study the relationship between centrality measures and the a priori ranking p of players. Centrality measures studied here exist uniquely for complete networks. For complete networks these measures are also easier to compute than for general incomplete networks.

Player i would want the networks to be as complete as possible, if we would append a term $N_i(s) = |\{j \neq i \mid s_{ij}s_{ji} > 0\}|$ to his utility function. So player i's utility would be an increasing function of the number of players j with whom i has both links L_{ij} and L_{ji}. One can check that the equilibrium given by equation 6 in the proof of Theorem 1 would remain an equilibrium after such a modification of utility functions.

Theorem 1. *For bilinear link formation games with $c = 0$ there exists an equilibrium with a complete network, iff*

$$\sum_{i \in N} p_i > (n-1)p_n.$$

Proof. Let s be an equilibrium with a complete network. Completeness of equilibrium network means that $s_{ij} > 0$ for all i, for all $j \neq i$. Since the opportunity cost of privacy is zero, $s_{ii} = 0$ for all i, since s_i is a best reply against s. The first order condition for player i is

$$s_{ji}p_j = v_i,\ \forall j \neq i, \tag{3}$$

where v_i is the equilibrium utility of player i. Keeping j fixed and taking the sum over $i \neq j$ in equation 3 we get

$$p_j + v_j = \sum_{i \in N} v_i,\ \forall j \in N, \tag{4}$$

since $\sum_{i \neq j} s_{ji} = 1$. Taking the sum over j on both sides of equation 4 we get $\sum_j v_j = \left(\sum_j p_j\right)/(n-1)$. Inserting this into equation 4 and rearranging gives us

$$v_j = \frac{1}{n-1}\sum_{i\in N} p_i - p_j,\ \forall j \in N. \tag{5}$$

Equation 5 implies $\sum_{i\in N} p_i > (n-1)p_n$, since $s_{ij}s_{ji}v_j > 0$ for all i, for all $j \neq i$.

Assume then that $\sum_{i\in N} p_i > (n-1)p_n$ holds, *i.e.* $v_j > 0$ for all j in equation 5. Inserting v_j from equation 5 into equation 3 we get

$$s_{ij} = \Big(\frac{1}{n-1}\sum_{k\in N} p_k - p_j\Big)p_i^{-1}. \tag{6}$$

Since $s_{ij} > 0$ and $\sum_{j\neq i} s_{ij} = 1$, equation 6 we have constructed an equilibrium s with a complete network. □

Remark 1. We assumed that $c = 0$ in Theorem 1. It is straightforward to verify that the equilibrium given by equation 6 remains an equilibrium if c satisfies $0 < c < \sum_j p_j/(n-1) - p_n$. In this case c is smaller than the equilibrium value v_j of any player j given by equation 5, and therefore every player j chooses $s_{ii} = 0$ when c is sufficiently small. This remark applies to Theorem 2 below as well.

Given a link formation game G, we say that an equilibrium network corresponding to equilibrium s is *maximally complete*, if the components of the network are all complete and there is no equilibrium s' with a coarser partition into components such that all the components are complete.

Theorem 1 gives the a condition for the existence of an equilibrium with a complete network. Even if there are no such equilibria, that is, the condition given in that theorem does not hold, there are always equilibria with a maximally complete network. We say that a subset A of natural numbers is an interval, if $i, j \in A$ implies $k \in A$ for all k such that $i < k < j$.

Proposition 1. *For bilinear link formation games there exists an equilibrium with a maximally complete network such that the components are intervals.*

Proof. Let $n_0 = n$, and let $n_1 \in N$ be the least integer such that

$$\sum_{i \geq n_1} p_i > (n - n_1 - 1)p_n.$$

By Theorem 1 there exists an equilibrium s^1 in the link formation game with player set $N_1 = \{n_1, \ldots, n\}$ such that the corresponding network is complete. Given that $n_k > 1$ and the corresponding interval N_k are defined $k \geq 1$, let $n_{k+1} \in N$ be the least integer such that

$$\sum_{n_{k+1} \leq i < n_k} p_i > (n_k - n_{k+1})p_{n_k - 1}.$$

Continue as long as $n_m = 1$ is reached. Then for each player set N_k there exists an equilibrium s^k with a complete network by Theorem 1. Let s be an equilibrium of G such that $s_i = s_i^k$ for all $i \in N_k$, $k = 1, \ldots, m$. Then s is an equilibrium with a maximally complete network. □

Denote by S the $n \times n$ (stochastic) matrix whose rows are the strategies $s_1, \ldots, s_n$. By equation 3 the row vector $p = (p_1, \ldots, p_n)$ and the vector of equilibrium values $v = (v_1, \ldots, v_n)$ satisfy

$$pS = (n-1)v. \tag{7}$$

Let us compute next the eigenvector centrality measure of S.

Theorem 2. *Suppose S is the stochastic matrix of equilibrium strategies given by equation 6 of a bilinear game of Theorem 1. Then the probability vector q satisfies $qS = q$, iff*

$$q_j = \frac{p_j - (n-1)p_j^2}{1 - (n-1)\sum_{i=1}^{n} p_i^2}, \; j \in N. \tag{8}$$

Proof. We may normalise the weight vector p so that $\sum_i p_i = 1$. This normalisation doesn't affect the equilibria of our link formation game. Now $qS = q$, iff there exist a strictly positive vector a such that $a_i p_i = q_i$ for all

$i \in N$. Therefore we have to solve $\sum_{i \neq j} s_{ij} a_i p_i = a_j p_j$ for all $j \in N$. By equation 6 this reduces to

$$\Big(\frac{1}{n-1} - p_j\Big) \sum_{i \neq j} a_i = a_j p_j, \; \forall j \in N, \tag{9}$$

since $\sum_i p_i = 1$. Rearranging the terms of equation 9 we get

$$\Big(\frac{1}{n-1}\Big) \sum_{i \neq j} a_i = p_j \sum_{i=1}^{n} a_i. \tag{10}$$

Rearranging the terms of equation 10 and multiplying both sides by p_j gives us

$$\Big(\sum_{i=1}^{n} a_i\Big)\big(1 - (n-1)p_j\big)p_j = a_j p_j. \tag{11}$$

Taking the sum with respect to j on both sides of equation 11 and simplifying gives us

$$\sum_{j=1}^{n} a_j p_j = \Big(\sum_{i=1}^{n} a_i\Big)\Big(1 - (n-1)\sum_{j=1}^{n} p_j^2\Big) = 1, \tag{12}$$

where the last equality holds because $\sum_{j=1}^{n} a_j p_j = 1$.

Equations 11 and 12 imply

$$a_j = \frac{1 - (n-1)p_j}{1 - (n-1)\sum_{i=1}^{n} p_i^2}, \; j \in N. \tag{13}$$

Therefore the fixed point equation $qS = q$ is satisfied by q such that

$$q_j = a_j p_j = \frac{p_j - (n-1)p_j^2}{1 - (n-1)\sum_{i=1}^{n} p_i^2}, \; j \in N.$$

This completes the proof. □

Remark 2. Note that Theorem 2 can easily be adapted by to cases in which no equilibrium has a complete network by Proposition 2. Given an equilibrium with a maximally complete network, each component N_k of that network has an equilibrium s^k. One can solve for the fixed point q^k corresponding to the stochastic matrix whose rows are strategies $s_i^k, i \in N_k$ just like in the proof of Theorem 2.

Let us study how the eigenvector centrality measure q of equation 8 and the indegree d are related to the original weights p of players. It turns out that player j is ranked higher than player i according to q and d, if player j was ranked *lower* than i according to p.

Proposition 2. *Let q be the fixed point of the stochastic matrix S of Theorem 2. A) If $p_j < p_i$, then $q_i < q_j$, for all $i, j \in N$. B) For indegrees we have that $p_j < p_i$ iff $d_i < d_j$, for all $i, j \in N$.*

Proof. A) Equation 8 implies immediately that $p_j < p_i$ implies $q_j < q_i$ iff $1/(n-1) > p_i + p_j$. If this holds for any i, j, then it must hold for $i = 1, j = 2$ since $p_1 < \cdots < p_n$. But in this case $p_1 + p_2 + \sum_{k>2} p_k = 1$ implies that $\sum_{k>2} p_k > 1 - 1/(n-1) = (n-2)/(n-1)$. Therefore $p_n > 1/(n-1)$, a contradiction with Theorems 1 and 2. Hence $p_j < p_i$ implies $q_i < q_j$, for all $i, j \in N$.

B) By equation 6 the indegree of player i is

$$d_i = \sum_{k \neq i} s_{ki} = \Big(\frac{1}{n-1} - p_i\Big) \sum_{k \neq i} \frac{1}{p_k}.$$

Then $d_i < d_j$ iff

$$\frac{1}{p_j} - (n-1)p_i \sum_{k \neq i} \frac{1}{p_k} < \frac{1}{p_i} - (n-1)p_j \sum_{k \neq j} \frac{1}{p_k}$$

iff

$$p_i - (n-1)p_i^2 p_j \sum_{k \neq i} \frac{1}{p_k} < p_j - (n-1)p_i p_j^2 \sum_{k \neq j} \frac{1}{p_k}$$

iff

$$p_i - p_j - (n-1)(p_i^2 - p_j^2) < (n-1)p_i p_j (p_i - p_j) \sum_{k \neq i,j} \frac{1}{p_k}$$

iff

$$(p_i - p_j)[1 - (n-1)(p_i + p_j)] < (p_i - p_j)(n-1)p_i p_j \sum_{k \neq i,j} \frac{1}{p_k}.$$

If $p_i < p_j$ would hold, then the last inequality would imply

$$1 - (n-1)(p_i + p_j) > (n-1)p_i p_j \sum_{k \neq i,j} \frac{1}{p_k}.$$

But this cannot hold since the right hand side is positive, and the left hand side is negative by the proof of case A) above. Hence $p_j < p_i$. $\square$

Let us compute next the Katz-Bonacich measure b of S.

Theorem 3. *Given $\alpha \in (0,1)$, the Katz-Bonacich measure b of the stochastic matrix S of Theorem 2 is given by*

$$b_i = \frac{p_i(\alpha v_i T + 1)}{p_i + \alpha v_i},$$

for all $i \in N$, where

$$T = \frac{n/(1-\alpha) - \sum_i v_i/(p_i + \alpha v_i)}{1/(n-1) - \alpha \sum_i v_i^2/(p_i + \alpha v_i)},$$

and $v_i = 1/(n-1) - p_i$ is the equilibrium value of player i given in equation 5.

Proof. The i'th column of the equation $b(I - \alpha S) = \mathbf{1}$ is

$$b_i - \alpha v_i \sum_{j \neq i} b_j/p_j = 1,$$

where $v_i = 1/(n-1) - p_i$ is the equilibrium value of player i. This is equivalent to

$$b_i + \alpha v_i b_i/p_i - \alpha v_i \sum_{j \in N} b_j/p_j = 1. \tag{14}$$

Taking the sum over i gives us

$$\sum_{i \in N} b_i - \alpha \Big[\Big(\sum_{i \in N} v_i \Big) \Big(\sum_{i \in N} b_i/p_i \Big) - \sum_{i \in N} v_i b_i/p_i \Big] = n. \tag{15}$$

The term in the square brackets is equal to $\sum_i (Sb)_i$. Since S is a stochastic matrix, $\sum_i b_i = \sum_i (Sb)_i$. Therefore the number in the square brackets equals $\sum_i b_i = n/(1-\alpha)$.

Let $T = \sum_{j \in N} b_j/p_j$. Then from equation 14 we get that

$$b_i/p_i = \frac{\alpha v_i T + 1}{p_i + \alpha v_i}. \tag{16}$$

Since b_i/p_i appears of both sides of equation 16, we must solve for T. Using equation 16 the term in the square brackets of equation 15 becomes

$$\frac{1}{n-1}T - \sum_{i \in N} \frac{v_i(\alpha v_i T + 1)}{p_i + \alpha v_i}, \tag{17}$$

because $\sum_i v_i = 1/(n-1)$. Using the fact that $\sum_i b_i = n/(1-\alpha)$, the term $T = \sum_i b_i/p_i$ in equation 17 can be solved:

$$T = \frac{n/(1-\alpha) - \sum_i v_i/(p_i + \alpha v_i)}{1/(n-1) - \alpha \sum_i v_i^2/(p_i + \alpha v_i)}. \tag{18}$$

Now the value of b_i can be solved from equation 16. □

Corollary 1. *Substituting $1/(n-1) - p_i$ for v_i the b_i values of Theorem 3 can be rewritten as*

$$b_i = \frac{p_i\big(\alpha(1-(n-1)p_i)T + n - 1\big)}{\alpha + (1-\alpha)(n-1)p_i}, \tag{19}$$

where

$$T = \frac{n(n-1) - (n-1)(1-\alpha)\sum_i \frac{1-(n-1)p_i}{\alpha+(1-\alpha)(n-1)p_i}}{(1-\alpha)\Big(1 - \alpha \sum_i \frac{(1-(n-1)p_i)^2}{\alpha+(1-\alpha)(n-1)p_i}\Big)}.$$

Let us study next how the vectors b and p are related. It turns out that the eigenvector centrality q and the Katz-Bonacich measure b give the same ordinal ranking of the players. Hence the ranking given by b is the opposite to the one given by p.

Proposition 3. *Suppose b is the Katz-Bonacich measure b of the stochastic matrix S of Theorem 3. Then $p_i < p_j$ implies $b_j < b_i$, for all $i, j \in N$.*

Proof. Given $\alpha > 0$ and the stochastic matrix S, the Katz-Bonacich measure b satisfies

$$bS = (b - \mathbf{1})/\alpha. \tag{20}$$

Therefore $b_i < b_j$ if and only if $(Sb)_i < (Sb)_j$. Let $j = 1$ and $i = 2$, and consider the first and second column of the matrix S. By equation 6 the

j'th element of the first column is $s_{j1} = (1/(n-1) - p_1)/p_j$ and the j'th element of the second column is $s_{j2} = (1/(n-1) - p_2)/p_j$, where $j \neq 1, 2$. Since $p_1 < \cdots < p_n$, we get that $s_{j2} < s_{j1}$.

Comparing $s_{12} = (1/(n-1) - p_2)/p_1$ and $s_{21} = (1/(n-1) - p_1)/p_2$ we get that $s_{12} < s_{21}$. To see this, note first that the quadratic function $(1/(n-1) - p)p$ is maximized at $p^* = 1/2(n-1)$. Hence $s_{12} < s_{21}$ holds, iff p_1 is closer to p^* than p_2. This holds if $p^* \leq p_1$ since $p_1 < p_2$. Consider next cases $p_1 < p^*$.

Note first that $p_2 \leq p^*$ would imply that $p_1 + p_2 < 1/(n-1)$, and therefore $p_n > 1/(n-1)$ since $\sum_i p_i = 1$. This is a contradiction because $p_n < 1/(n-1)$ by Theorem 1. Therefore we can assume $p_1 < p^* = 1/2(n-1) < p_2$. Let $p_1 = p^* - a_1$ and $p_2 = p^* + a_2$, $a_1, a_2 > 0$. If $a_2 \leq a_1$, then $p_1 + p_2 \leq 1/(n-1)$ and therefore $p_n \geq 1/(n-1)$, a contradiction. So $a_1 < a_2$, and hence $s_{12} < s_{21}$.

In the same way it can be shown that given any players i, j, k such that $i < j$ and $k \neq i, j$, the following inequalities hold: $s_{kj} < s_{ki}$, and $s_{ij} < s_{ji}$.

Given the Katz-Bonacich measure b, compute $(Sb)_1 - (Sb)_2$ by using equation 20. This gives us

$$\sum_{j \neq 1,2} (s_{j1} - s_{j2}) b_j - s_{12} b_1 + s_{21} b_2 = (b_1 - b_2)/\alpha.$$

It is impossible that $b_1 \leq b_2$, because in this case the left hand side of the equation would be strictly positive and the right hand side would be zero or negative. This follows since $s_{j1} - s_{j2}$ for all $j \neq 1, 2$, $s_{12} < s_{21}$, and b is a strictly positive vector.

In the same way it can shown that $b_j < b_i$ for all $i, j \in N$ such that $i < j$. Since $p_i < p_j$ iff $i < j$, we are done □

Let us now analyze concave link formation games. The utility function of player i is

$$u_i(s_i, s_{-i}) = \sum_{j \neq i} p_j s_{ij}^{\alpha} s_{ji}^{\beta} + c\Big(1 - \sum_{j \neq i} s_{ji}\Big), \tag{21}$$

where $\alpha, \beta > 0$, $\alpha + \beta < 1$, and $c > 0$. Since $\sum_j s_{ij} = 1$, $c(1 - \sum_{j \neq i} s_{ji})$ is the utility player i gets from acting alone. We do not assume that $\sum_i p_i = 1$ since we already have the normalization $c = 1$. It was shown in Example 2 in Salonen (2014) that there exists an interior equilibrium ($s_{ii}, s_{ij} > 0$ for all i, j) if

$$\alpha p < \left[\frac{1}{n-1}\right]^{1-\alpha-\beta}.$$

The equilibrium strategies are given by

$$s_{ij} = \alpha^{1/[1-\alpha-\beta]} \left[p_j^{1-\alpha} p_i^{\beta}\right]^{1/[(1-\alpha)^2-\beta^2]}, \forall i, j \in N. \tag{22}$$

Let again S be a stochastic matrix whose i'th row is the equilibrium strategy s_i of player $i \in N$. Now it turns out that the Katz-Bonancich index b ranks the players in the same way as p.

Proposition 4. *Let S be the stochastic matrix corresponding to the interior equilibrium of equation 22 of a concave link formation game. If b is the Katz-Bonacich measure of S, then $p_i < p_j$ implies $b_i < b_j$.*

Proof. It was noted in the proof of Proposition 3 that $b_i < b_j$, iff $(Sb)_i < (Sb)_j$. We employ a similar strategy of proof here as was used in the proof of Proposition 3: we show that $(Sb)_i < (Sb)_j$ holds if $i < j$, which in turn means that $p_i < p_j$. We prove that $(Sb)_1 < (Sb)_2$ holds, from which the result extends easily to arbitrary i, j with $i < j$.

The first two columns of S are $(0, s_{21}, s_{31}, \ldots, s_{n1})$ and $(s_{12}, 0, s_{32}, \ldots, s_{n2})$. The terms s_{21} and s_{12} are of the form: $s_{21} = A[p_1^{1-\alpha} p_2^{\beta}]^B$, and $s_{12} = A[p_2^{1-\alpha} p_1^{\beta}]^B$, where $A, B > 0$. Hence $s_{21} < s_{12}$, iff $p_1^{1-\alpha} p_2^{\beta} < p_2^{1-\alpha} p_1^{\beta}$, which holds because $p_1 < p_2$ and $\beta < 1 - \alpha$.

Comparing $s_{j1} = A[p_1^{1-\alpha} p_j^{\beta}]^B$ and $s_{j2} = A[p_2^{1-\alpha} p_j^{\beta}]^B$, $j > 2$, we observe immediately that $s_{j1} < s_{j2}$ since $p_1 < p_2$.

By using equation 20 the difference $(Sb)_1 - (Sb)_2$ is

$$\sum_{j \neq 1,2} (s_{j1} - s_{j2}) b_j - s_{12} b_1 + s_{21} b_2 = (b_1 - b_2)/\alpha.$$

Now the left hand side is negative since b is positive, and therefore $b_1 < b_2$. $\square$

Corollary 2. *Let S be the stochastic matrix corresponding to the interior equilibrium of equation 22 of a concave link formation game. Then the indegree d and the eigenvector centrality measure q of S give the players the same ranking as p.*

Proof. It was shown in the proof of Proposition 4 that $s_{ji} < s_{ij}$ and $s_{ki} < s_{kj}$ hold for columns i and j of S, when $i < j$ and $k \neq i, j$. Hence the indegrees of these columns satisfy $d_i < d_j$, which is the same ranking as given by p.

The eigenvector centrality q satisfies $Sq = q$, and therefore the equivalence $(Sq)_i < (Sq)_j$ iff $q_i < q_j$ holds automatically. By the previous paragraph, $\sum_{k \neq i,j}(s_{kj} - s_{ki})q_k > 0$ holds, and therefore $q_i < q_j$. □

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