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A Note on Nash Equilibrium and Fixed Point Theorems

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ABSTRACT

We give a proof of the existence of a Nash equilibrium for n -person normal form games when each player's utility function is continuous w.r.t. strategy profiles, and concave and differentiable w.r.t. his own strategy. The proof uses only elementary mathematical tools such as mathematical induction. We show that this equilibrium existence result is sufficiently general to imply the Brouwer Fixed Point Theorem. The Kakutani Fixed Point Theorem is obtained as a corollary by using standard techniques.

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1. Introduction

We give an elementary proof of the existence of a Nash equilibrium for n -person normal form games (Nash 1950, Nash 1951). The proof uses only basic mathematical tools such as mathematical induction, continuity of mappings, and compactness of subsets. This makes the proofs longer than necessary, but hopefully also more accessible to a wider audience.[1]

We show that the Brouwer Fixed Point Theorem (Brouwer 1912) follows from a Nash equilibrium existence theorem when each player's utility function is continuous w.r.t. strategy profiles, and concave and differentiable w.r.t. his own strategy. The Kakutani Fixed Point Theorem (Kakutani 1941) is obtained as a corollary by using standard techniques. This in turn implies the existence of a Nash equilibrium for normal form games when each player's utility functions satisfy certain conditions.

If utility functions are continuous on a compact convex subset of strategy profiles, and quasiconcave w.r.t. each player's own strategy, the best reply correspondence is nonempty valued, convex valued, and upper semicontinuous. Therefore the Kakutani Fixed Point Theorem implies the existence of a Nash equilibrium. Indeed, the conventional approach to prove the existence of equilibrium is to apply a suitable fixed point theorem such as the ones by Kakutani or Brouwer.

There has been certain kind of complementarity in the development of equilibrium theory and fixed point theory. For example, Kakutani was inspired by the Minimax Theorem of von Neumann (von Neumann (1928)) and wanted to find a simpler proof of it, and so he invented his fixed point theorem (see Park 1999, p. 198). For accessible surveys of fixed point problems and applications, see *e.g.* Border (1985), Ichiishi (1983), Park (1999).

Kakutani's theorem relies only on a few basic properties of a correspondence and its domain, and Brouwer's theorem has even more elementary structure. In game theoretic models, best reply mappings have a special product structure, and in addition, information about the utility function is missing from the formulation of Brouwer's or Kakutani's theorem. Therefore in some cases it might be possible to prove a fixed point theorem by first showing the existence of a Nash equilibrium, as we do in this paper.

2. The Results

Given a nonempty $X \subset \mathbb{R}^n$, a *correspondence* $f : X \rightrightarrows \mathbb{R}^m$ is a mapping such that $f(x) \subset \mathbb{R}^m$. A correspondence $f : X \rightrightarrows \mathbb{R}^m$ is *upper semicontinuous* (u.s.c.), if $x^n \longrightarrow \bar{x}$, $y^n \in f(x^n)$ and $y^n \longrightarrow \bar{y}$, imply that $\bar{y} \in f(\bar{x})$. A correspondence $f : X \rightrightarrows \mathbb{R}^m$ is *lower semicontinuous* (u.s.c.), if for each $y \in f(x)$, and for each sequence $\{x^n\}$ converging to x, there exists a sequence $\{y_n\}, y^n \in f(x^n)$, converging to y.

[1] Franklin (1983), p.15: "A private survey indicates that 96% of all mathematicians can state the Brouwer Fixed Point Theorem, but only 5% can prove it. Among mathematical economists, 95% can state it, but only 2% can prove it (and these are all ex-topologists).", and "While 96% of mathematicians can state the Brouwer Fixed Point Theorem, only 7% can state the Kakutani theorem."

Let $G = \{(u_i)_{i \in N}, (S_i)_{i \in N}; N\}$ be an n -person normal form game such that $N = \{1, \ldots, n\}$ is the set of players, the strategy set S_i is a nonempty compact convex subset of $\mathbb{R}^{n_i}$, and the utility function $u_i : \Pi_{i=1}^n S_i \longrightarrow \mathbb{R}$ is continuous and quasiconcave with respect to $s_i \in S_i$, given any $s_{-i} \in \Pi_{j \neq i} S_j$, for every $i \in N$. In Proposition 1 we assume that each u_i is concave and differentiable w.r.t. s_i.

A strategy profile $s \in S \equiv \Pi_{i=1}^n S_i$ is a *Nash equilibrium*, if for all $i \in N$ it holds that $u_i(\bar{s}) \geq u_i(s_i, \bar{s}_{-i})$, for all $s_i \in S_i$. Denote by $B_i(s)$ the set of strategies of i that maximize $u_i(s_i, s_{-i})$ given s_{-i}. The strategies in $B_i(s)$ are called player i's best replies against s_{-i}. Let $B(s) = \Pi_{i=1}^n B_i(s)$ be the product of best replies. Then $\bar{s}$ is a Nash equilibrium, if and only if $\bar{s} \in B(\bar{s})$.

We will prove the existence of a Nash equilibrium in a special case when the strategy sets have a simple product form: $S_i = [0,1]^{n_i}$, for some natural number $n_i \geq 1$.

Proposition 1. *Let G be a normal form game such that strategy sets are of the form $S_i = [0,1]^{n_i}, \in N$, and utility functions are continuous on $\Pi_{i=1}^n S_i$, and concave and differentiable w.r.t. each player's own strategy $s_i \in S_i$, given any $s_{-i} \in \Pi_{j \neq i} S_j$. Then there exists a Nash equilibrium.*

Proof. Let G be an n -person game such that $S_i = [0,1]$, for each $i \in N$. Given any strategies $s_2, \ldots, s_n$ of players $2, \ldots, n$, player 1 has a best reply against s_{-1}. We will show in *Step 1.* that there exists an equilibrium each player $i = 1, \ldots, n$ has a strategy set $[0,1]$. In *Step 2.* we show that an equilibrium exists when strategy sets are of the form $[0,1]^{n_i}$.

Step 1.

Suppose that given any $s_1 \in S_1$, the $n-1$ -player game played by players $i = 2, \ldots, n$ has a Nash equilibrium $s_{-1}(s_1)$. Since utility functions are continuous, the set of Nash equilibria of this $n-1$ -player game is compact for each s_1. Denote by $E(s_1)$ the set of these equilibria. Then $E : S_1 \rightrightarrows S_{-1}$ is u.s.c. To see this, let $\{s_1^k\}$ be a sequence converging to s_1. Take any $s_{-1}^k \in E(s_1^k)$, $k = 1, 2, \ldots$. Then $\{s_{-1}^k\}$ has a convergent subsequence because strategy sets are compact, and we may assume w.l.o.g. that this sequence itself converges to s_{-i}. By continuity of utility functions, $s_{-i} \in E(s_1)$. Therefore E is u.s.c. and it has a closed graph $gr(E) = \{(s_1, s_{-1}) \mid s_{-1} \in E(s_1), s_1 \in [0,1]\}$. The graph $gr(E)$ is compact because it is closed in $S_1 \times \cdots \times S_n$.

For each $s_1 > 0$, let $E^+(s_1) \subset E(s_1)$ denote the subset of those equilibria $s_{-1} \in E(s_1)$ for which there exists a increasing sequence s_1^k converging to s_1, and a sequence $s_{-1}^k \in E(s_1^k)$ converging to s_{-1}. Clearly, $E^+(s_1)$ is nonempty and closed for each $s_1 > 0$.

Suppose $s_1 = 1$ is not a best reply against any $s_{-1} \in E(1)$. Then every best reply $r_1(s_{-1}) \in B_1(s_{-1})$ of player 1 against any $s_{-1} \in E(1)$ satisfies $r_1(s_{-1}) < 1$. It is not difficult to see that there exists $\delta \in (0,1)$ such that for all $t \in (\delta, 1)$, and all $s_{-1} \in E^+(t) \subset E(t)$, every best reply $r_1(s_{-1})$ satisfies $r_1(s_{-1}) < t$. Let δ^* be the infimum of such numbers δ.

There are two cases:

a) at δ^*, there exists $s_{-1} \in E^+(\delta^*)$ and a best reply $r_1(s_{-1})$ such that $r_1(s_{-1}) \geq \delta^*$ (this case covers the situation $\delta^* = 0$);

b) at δ^*, for every $s_{-1} \in E^+(\delta^*)$ and any best reply $r_1(s_{-1})$ it holds that $r_1(s_{-1}) < \delta^*$.

Case a). By u.s.c. of best replies, for $s_{-1} \in E^+(\delta^*)$ such that $r_1(s_{-1}) \geq \delta^*$, there exists also another best reply $r_1'(s_{-1}) \leq \delta^*$. Since best reply correspondences have convex values, the strategy δ^* is also a best reply against s_{-1}.

Case b). By definition of δ^*, arbitrarily close to δ^* there are numbers $t < \delta^*$ such that there exists $s_{-1} \in E^+(t)$ such that for some best reply $r_1(s_{-1}) \in B_1(s_{-1})$ it holds that $r_1(s_{-1}) \geq t$. Let t^k be an increasing sequence converging to δ^* such that there exists $s_{-1}^k \in E^+(t^k)$ such that for some best reply $r_1(s_{-1}^k)$ it holds that $r_1^k(s_{-1}) \geq t^k$. Then $\{s_{-1}^k\}$ has a convergent sequence and without loss of generality this sequence itself converges to s_{-1}^*.

Now $s_{-1}^* \in E^+(\delta^*)$, since t^k be an increasing sequence converging to δ^*, and $s_{-1}^k \in E^+(t^k)$ converges to $s_{-1}^* \in E^+(\delta^*)$. By u.s.c. of best replies, there is a best reply $r_1(s_{-1}^*)$ satisfying $r_1(s_{-1}^*) \geq \delta^*$, a contradiction with the Case b).

Therefore there exists a Nash equilibrium, when $S_i = [0,1]$ for each $i = 1, \ldots, n$.

Step 2.

Suppose a Nash equilibrium exists in a game G^- when $S_i = [0,1]^{n_i}$, $n_i \geq 1$. Choose any player i, and replace his strategy set $[0,1]^{n_i}$ by a strategy set $[0,1]^{n_i+1}$. Without loss of generality, let $i = 1$. Let G be a game such that $S_1 = [0,1]^{n_1+1}$, and $S_j = [0,1]^{n_j}$, for $j \neq i$. Denote player 1's strategies by $x = (x_1, s_1)$, where $s_1 \in [0,1]^{n_1}$, and $x_1 \in [0,1]$.

Fix any $x_1 \in [0,1]$, let player 1 choose freely from $s_1 \in [0,1]^{n_1}$, and let other players j choose freely from $[0,1]^{n_j}$. Then this game G^- has a Nash equilibrium $s(x_1)$ by induction assumption. Denote by $E(x_1)$ the set of all Nash equilibria given x_1. By a similar argument than in the one-dimensional case of *Step 1*, the correspondence $E : [0,1] \rightrightarrows [0,1]^{n_1} \times \cdots \times [0,1]^{n_n}$ is u.s.c. and hence it has a closed graph.

The rest of the argument is analogous to the one given in the one-dimensional case in *Step 1.* Define $E^+(x_1)$ in the same way as $E^+(s_1)$ was defined in *Step 1.* By a similar argument, one can show that there exists $\delta^* \in [0,1]$ such that $s(\delta^*)$ is a Nash equilibrium in the game G^-, and δ^* maximizes the utility function $u_1((x_1, s_1(\delta^*)), s_{-1}(\delta^*))$ of player 1.

We have found a strategy profile $((\delta^*, s_1(\delta^*)), s_{-1}(\delta^*))$ such that

1) $s_j(\delta^*)$ is a best reply against $s_{-j}(\delta^*)$ for every $j \neq 1$;
2) $s_1(\delta^*)$ is a best reply against $(\delta^*, s_{-1}(\delta^*))$;
3) δ^* is a best reply against $(s_1(\delta^*), s_{-1}(\delta^*))$.

Since each u_i is concave and differentiable w.r.t. s_i, cases 2) and 3) mean that the first order partial derivatives of u_1 equal zero (or are nonpositive in case of a boundary solution). Therefore $(\delta^*, s_1(\delta^*))$ is a best reply against $s_{-1}(\delta^*)$, and $((\delta^*, s_1(\delta^*)), s_{-1}(\delta^*))$ is a Nash equilibrium of the game G. □

Remark 1. Note that the assumption that each u_i is concave and differentiable w.r.t. s_i was needed only in the last paragraph of the proof. It might not be easy to get rid of the differentiability assumption. Consider a function $u(x_1, x_2) = \min\{x_1, x_2\}$ on $[0,1]^2$. The function u is concave w.r.t. (x_1, x_2), but not differentiable on the diagonal. The maximum is at $(1,1)$. If the variables x_1 and x_2 are chosen by two agents independently, both agents having the utility function u, then any point on the diagonal is a Nash equilibrium.

We will next show that Proposition 1 implies the Brouwer Fixed Point Theorem.

Theorem 1 (Brouwer Fixed Point Theorem)**.** *If $X \subset \mathbb{R}^m$ is a nonempty compact and convex subset of $\mathbb{R}^m$, and $f : X \longrightarrow X$ is a continuous function, then f has fixed point.*

Proof. We will first prove the Brouwer Fixed Point Theorem in the special case when $X = [0,1]^m$.

Given a continuous function $g : [0,1]^m \longrightarrow [0,1]^m$, define a two-person game as follows. Let $S_1 = S_2 = [0,1]^m$, and define $u_1(s_1, s_2) = -[s_{11} - g_1(s_2)]^2 - \cdots - [s_{1m} - g_m(s_2)]^2$, and $u_2(s_1, s_2) = -[s_{21} - s_{11}]^2 - \cdots - [s_{2m} - s_{1m}]^2$, where $s_i = (s_{i1}, \ldots, s_{im})$, and $g(s_2) = (g_1(s_2), \ldots, g_m(s_2))$.

The functions u_i are continuous, and concave and differentiable w.r.t. s_i. By Proposition 1, the game G has a Nash equilibrium $(\bar{s}_1, \bar{s}_2)$. Therefore $\bar{s}_1 = g(\bar{s}_2)$, and $\bar{s}_2 = \bar{s}_1$. Hence $\bar{s}_1$ is a fixed point of g.

Clearly a fixed point exists also for a continuous $g : [-K, K]^m \longrightarrow [-K, K]^m$, given any $K > 0$.

Consider next the general case $f : X \longrightarrow X$, where $X \subset \mathbb{R}^m$ is any compact and convex set. We may assume w.l.o.g. that the origin $\mathbf{0}$ is an interior point of X, and that $X \subset [-K, K]^m$. For each $x \in [-K, K]^m$, let $t(x) > 0$ be the greatest number such that $t(x)x \in X$, if $x \in [-K, K]^m \setminus X$, and $t(x) = x$, if $x \in X$. Define $g : [-K, K]^m \longrightarrow [-K, K]^m$ by $g(x) = f(t(x))$.

Then g has a fixed point since g is continuous. All fixed points of g are in X, and they are also fixed points of f. □

Remark 2. The trick in the proof of Theorem 1 how to replace a compact and convex set $[-K, K]^m$ by more general compact and convex sets X is well-known, and it is given in the proof for completeness only without any claim of originality. There are other well-known methods that would work equally well.

Corollary 1 (Kakutani Fixed Point Theorem)**.** *If $X \subset \mathbb{R}^n$ is a nonempty compact and convex subset of $\mathbb{R}^n$, and $f : X \rightrightarrows X$ is a upper semicontinuous correspondence, then f has fixed point.*

Proof. Let $gr(f) = \{x, f(x) \mid x \in X\}$ be the graph of f. The graph is closed since f is u.s.c. Given $(x, y) \in gr(f)$, and $\varepsilon > 0$, let $B((x, y), \varepsilon) \subset X \times X$ be the open ε -ball around (x, y) in $X \times X$. Let

$$V(gr(f), \varepsilon) = \bigcup_{(x,y) \in gr(f)} B((x, y), \varepsilon).$$

be the open ε neighbourhood around $gr(f)$. Then the correspondence $F^\varepsilon : X \rightrightarrows X$ defined by $F^\varepsilon(x) = \{y \in X \mid (x, y) \in V(gr(f), \varepsilon)\}$ is lower semicontinuous.

By Michael's Selection Theorem (Michael 1956), to each ε, there exists a continuous function $f^\varepsilon : X \longrightarrow X$ such that $(x, f^\varepsilon(x)) \in gr(F^\varepsilon)$. By the Brouwer Fixed Point Theorem, f^ε has a fixed point x^ε. The set $\{(x^\varepsilon, x^\varepsilon)\}_\varepsilon$ has a cluster point $(\bar{x}, \bar{x})$ in $gr(f)$, and $\bar{x}$ is a fixed point of f. □

Remark 3. The trick of using Michael's Selection Theorem to prove Kakutani Fixed Point Theorem is well-known and shown here for completeness only.

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