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ABSTRACT

We analyze the determination of a value maximizing dividend policy for a broad class of cash flow processes modelled as spectrally negative jump diffusions. We extend previous results based on continuous diffusion models and characterize the value of the optimal dividend policy explicitly. Utilizing this result, we also characterize explicitly the values as well as the optimal dividend thresholds for a class of associated optimal stopping and sequential impulse control problems. Our results indicate that both the value as well as the marginal value of the optimal policy are increasing functions of policy flexibility in the discontinuous setting as well.

JEL Classification: C61, G35

Keywords: dividend optimization, downside risk, impulse control, jump diffusion, optimal stopping, singular stochastic control

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1 Introduction

Dividends are one way in which firms distribute some of their profits to the shareholders. The dividend policy of a firm should specify the rules according to which dividends are paid out – most importantly, the size of a dividend payment and its timing. In addition to the most obvious example of a firm paying dividends to its shareholders, also bonuses given to customers by an insurance undertaking can be viewed as dividend distribution.

Mathematically, the problem of determining the optimal dividend distribution policy in absence of transaction costs can be formulated as a singular stochastic control problem. The singular controls can usually be expressed in terms of the local times of the underlying value process, i.e. they correspond to so-called barrier strategies, in which all retained earnings exceeding a given level are distributed to shareholders. If there is a fixed cost associated with a transaction, the optimal dividend policy takes the form of an impulse control consisting of lump sum dividends distributed at discrete moments of time. In Alvarez and Virtanen (2006) it is shown that in a diffusion model, under relatively general conditions, the value of the impulse control problem is always dominated by the value of the singular control problem. This is quite intuitive, as the singular case is the one allowing the most flexible dividend policies and, in fact, an impulse control is an admissible dividend policy for the singular control problem as well. It could be argued that an impulse control corresponds more closely to actual reality. However, even if this argument would be accepted, we can still extract much useful information from the solution of the singular problem – besides, despite the dearth of closed form expressions for the local time process itself, the decision rule implied by a local time control is intuitive and casts light on the required rate of return in the associated discrete setting as well.

In modeling the stochastic dynamics of the cash flow, continuous processes have been more popular than processes with discontinuities – largely due to their mathematically more convenient properties. The dividend problem has been considered, among many others, in Taksar (2000) and Gerber and Shiu

(2004) (in an insurance context utilizing a diffusion approximation of the surplus process).

However, from a risk management point of view the assumption of path continuity neglects the downside risk, the possibility of an instantaneous deterioration in the value of the reservoir of assets. This risk can be significant, as is evidenced, for example, by the effects of stock market crashes which may cause large instantaneous drops in the asset values. It is also well known that there is an asymmetry in the response of the market to new information: reactions to bad news are considerably stronger than reactions to good news (this is the celebrated *bad news principle* originally introduced in Bernanke (1983)). Moreover, in insurance applications most quantities of interest are naturally jump processes (think of the claims process). These considerations have led to a growing interest in models with stochastic dynamics allowing jumps, and in recent years, several results have been obtained. The most popular choice of dynamics appears to be the Lévy process in one form or another (the reason being again, of course, the relative tractability of this setting in comparison with more general Markovian dynamics). With regard to our main object of interest in this study, the *optimal dividend problem*, we mention particularly Perry and Stadje (2000) and Bar-Ilan et al (2004), where a stochastic cash management model with dynamics characterized by a finite activity Lévy process is considered, and the recent papers by Avram et al (2006) and Kyprianou and Palmowski (2006), where the authors investigate the optimal dividend policies under dynamics given by a spectrally negative Lévy process, using the fluctuation theory of Lévy processes, which for the spectrally negative case takes on a particularly simple form. *Optimal stopping* and *option pricing applications* in the context of (general and one-sided) Lévy processes have been by now studied extensively in literature; for a taste, see Gerber and Landry (1998), Gerber and Shiu (1998), Alili and Kyprianou (2005), Mordecki (2002a), Mordecki (2002b) and Mordecki and Salminen (2006). In a recent preprint Bayraktar and Egami (2006) consider optimization of venture capital investments in a jump diffusion model. Transforms applicable to solving many econometric and valuation problems for *affine* jump diffusions have been considered in Duffie et al (2000).

In light of this increased interest on Lévy models and recognition of the importance of downside risk, it is to some extent surprising that the possibilities suggested by the classical theory of diffusions and minimal excessive maps seem to have largely been neglected in the studies based on one-dimensional jump diffusion models (for an exception, see Mordecki and Salminen (2006)). It is namely the case that several of the results derived for continuous diffusions via the classical theory in Alvarez (2001) can be shown to hold true for spectrally negative jump diffusions (modulo some conditions), as has been demonstrated for optimal stopping problems in Alvarez and Rakkolainen (2006). These results allow in a relatively broad setting the reduction of the dynamic problem to static optimization.

Motivated by the previous considerations, we consider in this study the optimal distribution of dividends when the retained earnings from which dividends are paid out evolves as a spectrally negative jump diffusion with geometric (i.e. proportional) jumps. The jumps of the process reflect the downside risk. We shall state a set of reasonably general conditions under which the optimal singular control is a barrier strategy, possibly with an exceptional initial lump sum dividend in case the initial state is above the optimal barrier. Extending the results of Alvarez (2001), we prove that under these conditions the value of the optimal singular dividend policy has a representation in terms of the minimal increasing r-excessive mapping with respect to the underlying reserve process, that is, in terms of the increasing fundamental solution of an associated integro-differential equation characterizing the smooth minimal r-harmonic maps. This representation allows the reduction of the dynamic problem into a static minimization problem. We also show that under our assumptions both the associated optimal liquidation problem as well as the associated discrete dividend optimization (i.e. impulse control) problem are solvable in terms of the minimal increasing r-excessive map. We also extend the sandwiching result of Alvarez and Rakkolainen (2006) and demonstrate that the value of the considered stochastic control problems of the underlying discontinuous cash flow dynamics can be sandwiched between the values of two associated stochastic control problems based on a continuous cash flow process in the present setting as well.

Our study proceeds as follows. In section two we specify the stochastic dynamics of our jump diffusion, state our main assumptions on the parameters of the process and the associated integro-differential equation and present the mathematical formulation of the dividend control problem in absence of transaction costs. Section three gathers some auxiliary results, including a crucial uniqueness and existence theorem. These results are then used in the next section where our main theorem, the representation of the value of the singular dividend control problem in terms of the minimal r-excessive map is stated and proved. Furthermore, some interesting corollaries extending the results obtained for continuous diffusions in Alvarez and Virtanen (2006) are given. In particular, a representation theorem for the associated optimal stopping problem is proved. In section five we turn our attention to the determination of the optimal dividend control in presence of a fixed transaction cost, give the definition of the ensuing impulse control problem and obtain as corollaries of our main theorem the results that both the impulse control problem and its associated optimal stopping problem are solvable in terms of the minimal increasing r-excessive map as well. In the next section we illustrate our general results with an explicit mean-reverting model, the logistic Lévy diffusion. In particular, we demonstrate how our results allow us to evaluate the impact that the shape of the jump size distribution has on the optimal policies. Finally, concluding comments are presented in section seven.

2 Basic Setup and Assumptions

Our main objective in this study is to investigate the combined impact of continuous risk as well as potentially discontinuous downside risk on the rational dividend policy and on the value of a risk neutral firm. In order to accomplish this task, we assume that the reservoir of retained earnings from which dividends are paid out evolves in the absence of interventions according to a Lévy diffusion whose dynamics are governed by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - \int_{(0,1)} X_t z \tilde{N}(dt, dz), \tag{1}$$

$X_0 = x > 0$, where $\tilde{N}(dt, dz)$ is a compensated compound Poisson process (and thus a martingale) with the associated Lévy measure $\nu = \lambda\mathfrak{m}$, and $\mathfrak{m}$ is the jump size distribution, which is assumed to have a density $f_{\mathfrak{m}} \in C((0,1))$. It is worth noting that if $\tilde{N}$ is just a compound Poisson process (hence not a martingale), we can add to the drift and subtract from the jump component a suitable compensator to obtain a stochastic differential equation of form (1). The drift coefficient $\mu(x)$ and the volatility coefficient $\sigma(x) > 0$ are assumed to satisfy the usual conditions for the existence of a unique adapted cádlág solution $X \in L^2(\mathbb{P}_x)$ of (1) (Lipschitz continuity and at most linear growth, see Øksendal and Sulem (2005) Theorem 1.19). In addition, we assume that $\mu(x)$ is continuously differentiable. Furthermore, we assume that the standard absence of speculative bubbles condition is met and consider only cash flow processes with finite expected cumulative present values. That is, we analyze processes X satisfying the inequality

$$\mathbb{E}_x \int_0^{\zeta} e^{-rs} X_s ds < \infty, \tag{2}$$

where $\zeta \in (0, \infty]$ denotes the lifetime of the process and $r > 0$ denotes the constant discount rate. For notational convenience, we denote the space of cash flows with finite expected present value by $\mathcal{L}^1$. The state space of $X = \{X_t\}$ is $I = [0, \infty)$, where the boundaries 0 and ∞ are unattainable, and the solvency set $\mathbb{S} = (0, \infty)$. The lifetime of the process is then equal to the first exit time of X from $\mathbb{S}$, τ_0: as the boundaries of the state space are unattainable, $\tau_0 = \infty$. The negative coefficient of the jump part in (1) implies that the process is spectrally negative: it can decrease discontinuously but increases only continuously. This spectral negativity will play a crucial role in our analysis. The following assumption is made:

A1. X is *regular* in the sense that for all $x, y \in I$ it holds that $\mathbb{P}_x(\tau_y < \infty) = 1$, where $\tau_y = \inf\{t > 0 : X_t \geq y\}$.

Assumption *A1* ascertains the a.s. finiteness of the first exit time τ_u of X from any interval of form $(0, u)$ with $u < \infty$. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the natural filtration $\mathbb{F} = \{\sigma(X_s : s \leq t)\}_{t \in \mathbb{R}_+}$.

The integro-differential operator coinciding with the infinitesimal generator of X is defined for sufficiently smooth mappings $f(x)$ by

$$\begin{aligned}(\mathcal{G}f)(x) = &\frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) + \\ &+ \lambda \int_{(0,1)} \{f(x - xz) - f(x) + xzf'(x)\}\mathfrak{m}(dz).\end{aligned} \tag{3}$$

We will make use of the notation $\mathcal{G}_r u = \mathcal{G}u - ru$ and assume

A2. There exists an increasing C^2 solution ψ of $\mathcal{G}_r\psi = 0$ such that $\psi(0) = 0$.

By virtue of Lemma 3.2 in Alvarez and Rakkolainen (2006) such increasing solution is unique up to a multiplicative constant. It is worth noting that the smoothness of the solution may present some problems in the general setting – we mention that in Chan and Kyprianou (2006) it is shown that in the case of an (arithmetic) Lévy process with a nonzero Gaussian coefficient, the solution (which in this particular case is called *r-scale function*) belongs to $C^2(I)$. We define a differential operator associated with $\mathcal{G}_r$ for $f \in C^2(I)$ by

$$(\tilde{\mathcal{A}}_\theta f)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \tilde{\mu}(x)f'(x) - \theta f(x), \tag{4}$$

where $\theta \in (0, \infty)$ and

$$\tilde{\mu}(x) = \mu(x) + \lambda x \cdot \int_{(0,1)} z\mathfrak{m}(dz) = \mu(x) + \lambda \bar{z} x. \tag{5}$$

This operator is related to the continuous diffusion $\tilde{X}$ given by

$$d\tilde{X}_t = \tilde{\mu}(\tilde{X}_t)dt + \sigma(\tilde{X}_t)dW_t. \tag{6}$$

Along the lines of our previous notation, we denote as $\tilde{\psi}_\theta(x)$ the increasing fundamental solution of the ordinary linear second order differential equation $(\tilde{\mathcal{A}}_\theta u)(x) = 0$ (for a comprehensive characterization of these mappings, see Borodin and Salminen (2002), p. 33). As we will later demonstrate, the mappings $\tilde{\psi}_r(x)$ and $\tilde{\psi}_{r+\lambda}(x)$ can be applied for providing useful inequalities concerning the considered stochastic control problems.

Having characterized the underlying stochastic cash flow dynamics (1) in the absence of interventions we now denote the controlled cash flow dynamics as X_t^D and assume that it is characterized by the stochastic differential equation

$$dX_t^D = \mu(X_t^D)dt + \sigma(X_t^D)dW_t - \int_{(0,1)} X_t^D z\tilde{N}(dt, dz) - dD_t, \tag{7}$$

$X_{0-}^D = x$, where D denotes the implemented dividend policy. As usually, we call a dividend payout strategy *admissible* if it is non-negative, adapted, cádlág, and non-decreasing, and denote the class of admissible policies as $\mathcal{A}$. Under our assumptions X^D is a semimartingale (being a Markov process generated by a pseudodifferential operator, see Jacob and Schilling (2001)). In light of this characterization, our objective is to consider the determination of an admissible payout policy maximizing the expected cumulative present value of the dividend flow. Formally, our objective is to solve the cash flow management problem

$$V_S(x) = \sup_{D \in \mathcal{A}} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s, \tag{8}$$

where $\tau_0^D = \inf\{t > 0 : X_t^D \leq 0\}$ denotes the lifetime of the controlled reserve process X^D. It is worth emphasizing that in our model liquidation is always the result of a control action (and, thus, *endogenous*), as the assumed boundary behavior of X implies that exogenous liquidation in finite time is not possible.

As was pointed out in Alvarez and Virtanen (2006), the singular control setting is the one allowing the greatest flexibility in dividend policies, as single optimal stopping rules and discrete impulse policies (sequential stopping) are in fact admissible controls (belong to $\mathcal{A}$). In light of this observation, we define the optimal stopping problem associated to the singular stochastic control problem (8) as

$$V_{\mathrm{OSP}}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r\tau} X_\tau \right], \tag{9}$$

where $\mathcal{T}$ is the set of all $\mathbb{F}$-stopping times. Note that the valuation in (9) is perpetual since as was mentioned above, the underlying reserves cannot vanish nor explode in finite time.

3 Some Auxiliary Results

Before proceeding in our analysis of the considered dividend optimization problem in a general setting, we first define the net appreciation rate $\rho : I \to \mathbb{R}$ of the stock X as $\rho(x) = \mu(x) - rx$ and assume throughout this study that it has a finite expected cumulative present value, that is, that $\rho \in \mathcal{L}^1$. As will turn out later in our analysis, this mapping plays a key role in the determination of the optimal payout policy and its value. An interesting result based on this mapping is now summarized in the following.

Lemma 3.1. *For all $x \in I$ it holds that*

$$V_S(x) \leq x + \sup_{D \in \mathcal{A}} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} \rho(X_s^D) ds. \tag{10}$$

Especially, if $\rho(x) \leq 0$ for all $x \in I$ then the optimal strategy is to liquidate the corporation immediately and pay out the entire reserve instantaneously. In that case the value of the optimal dividend policy reads as $V_S(x) = x$ for all $x \in I$. Moreover, $V_{OSP}(x) = x$ for all $x \in I$ as well.

Proof. Applying the generalized Itô theorem to the identity mapping $x \mapsto x$ yields

$$\mathbb{E}_x \left[e^{-r\tau_N} X_{\tau_N}^D \right] = x + \mathbb{E}_x \int_0^{\tau_N} e^{-rs} \rho(X_s^D) ds - \mathbb{E}_x \int_0^{\tau_N} e^{-rs} dD_s,$$

where $\tau_N = N \wedge \tau_0^D \wedge \inf\{t \geq 0 : X_t^D > N\}$ is an increasing sequence of almost surely finite stopping times tending towards τ_0^D. Reordering terms, invoking the nonnegativity of the controlled jump-diffusion, and letting $N \to \infty$ yields by monotone convergence inequality (10). The optimality of instantaneous liquidation is then clear in light of (10). □

Lemma 3.1 characterizes the circumstances under which the so-called *take the money and run* policy (i.e. immediate liquidation of the company) is optimal. As intuitively is clear, waiting is suboptimal whenever the value of the reserves depreciates at all states and subsequently no intertemporal gains may be accrued by postponing the payout decision into the future. An interesting implication of the findings of Lemma 3.1 is that if the net appreciation rate has a global maximum at $\hat{x} = \text{argmax}\{\rho(x)\}$, then

$$x \leq V_S(x) \leq x + \frac{\rho(\hat{x})}{r}$$

for all $x \in I$. Thus, as long as the net appreciation rate is bounded, the value of the optimal policy can grow at most at a linear rate for large reservoirs.

Lemma 3.1 characterizes the optimal policy only in the extreme case of instantaneous liquidation. However, in order to characterize the optimal dividend payout policy in a more general setting more analysis is naturally needed. Before proceeding in our analysis we first define the continuously differentiable mappings $H : I^2 \mapsto \mathbb{R}$ and $\tilde{H}_\theta : I^2 \mapsto \mathbb{R}$ as

$$H(x,y) = \begin{cases} x - y + \frac{\psi(y)}{\psi'(y)} & x \geq y \\ \frac{\psi(x)}{\psi'(y)} & x < y \end{cases} \tag{11}$$

and

$$\tilde{H}_\theta(x,y) = \begin{cases} x - y + \frac{\tilde{\psi}_\theta(y)}{\tilde{\psi}'_\theta(y)} & x \geq y \\ \frac{\tilde{\psi}_\theta(x)}{\tilde{\psi}'_\theta(y)} & x < y. \end{cases} \tag{12}$$

It is worth noticing that for a given fixed $y \in I$ the function $x \mapsto H(x,y)$ satisfies the variational equalities

$$\begin{aligned} (\mathcal{G}_r H)(x,y) &= 0, \quad x < y \\ \partial_x H(x,y) &= 1, \quad x \geq y. \end{aligned}$$

Analogously, for a given fixed $y \in I$ the function $x \mapsto \tilde{H}_\theta(x,y)$ satisfies the variational equalities

$$\begin{aligned} (\tilde{\mathcal{A}}_\theta \tilde{H}_\theta)(x,y) &= 0, \quad x < y \\ \partial_x \tilde{H}_\theta(x,y) &= 1, \quad x \geq y. \end{aligned}$$

As we will later observe, these functions can be applied for solving the variational inequalities $\max\{(\mathcal{G}_r v)(x), (x-c) - v(x)\} = 0$, $\max\{(\mathcal{G}_r v)(x), 1 - v'(x)\} = 0$, $\max\{(\tilde{\mathcal{A}}_\theta u)(x), (x-c) - u(x)\} = 0$, and $\max\{(\tilde{\mathcal{A}}_\theta u)(x), 1 - u'(x)\} = 0$ associated to the considered singular control and optimal stopping problems. We can now establish the following result characterizing how the values of the mapping $\tilde{H}_\theta(x,y)$ defined with respect to the minimal increasing r-harmonic function for the continuous diffusion $\tilde{X}$ can be applied for bounding the values of the mapping $H(x,y)$ defined with respect to the jump-diffusion X.

Lemma 3.2. *For all $x, y \in I$ it holds that $\tilde{H}_{r+\lambda}(x,y) \leq H(x,y) \leq \tilde{H}_r(x,y)$. Consequently, $\sup_{y\in I} \tilde{H}_{r+\lambda}(x,y) \leq \sup_{y\in I} H(x,y) \leq \sup_{y\in I} \tilde{H}_r(x,y)$ provided that the supremum exists.*

Proof. As was established in Theorem 4.1 of Alvarez and Rakkolainen (2006) we have that

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \leq \frac{\psi(x)}{\psi(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)} \tag{13}$$

for all $0 < x \leq y < \infty$. This inequality and the fundamental theorem of integral calculus in turn implies that

$$\int_x^y \frac{\tilde{\psi}'_{r+\lambda}(t)}{\tilde{\psi}_{r+\lambda}(y)} dt \geq \int_x^y \frac{\psi'(t)}{\psi(y)} dt \geq \int_x^y \frac{\tilde{\psi}'_r(t)}{\tilde{\psi}_r(y)} dt.$$

Applying now the mean value theorem and letting $x \uparrow y$ then shows that

$$\frac{\tilde{\psi}'_{r+\lambda}(y)}{\tilde{\psi}_{r+\lambda}(y)} \geq \frac{\psi'(y)}{\psi(y)} \geq \frac{\tilde{\psi}'_r(y)}{\tilde{\psi}_r(y)} \tag{14}$$

for all $y \in I$. Noticing now that

$$\frac{\psi(x)}{\psi'(y)} = \frac{\psi(x)}{\psi(y)} \frac{\psi(y)}{\psi'(y)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y)} \frac{\tilde{\psi}_r(y)}{\tilde{\psi}'_r(y)} = \frac{\tilde{\psi}_r(x)}{\tilde{\psi}'_r(y)}$$

and

$$\frac{\psi(x)}{\psi'(y)} = \frac{\psi(x)}{\psi(y)} \frac{\psi(y)}{\psi'(y)} \geq \frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y)} \frac{\tilde{\psi}_{r+\lambda}(y)}{\tilde{\psi}'_{r+\lambda}(y)} = \frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}'_{r+\lambda}(y)}$$

then completes the proof of the alleged result. □

Lemma 3.2 states two interesting inequalities characterizing how the value of the function $H(x,y)$ can be sandwiched between the values $\tilde{H}_{r+\lambda}(x,y)$ and $\tilde{H}_r(x,y)$. This observation is of interest since it demonstrates that the solutions of the associated variational inequalities are ordered. As we will later observe, these functions are closely related to the values of the optimal dividend policies in the considered three different cases. Two interesting implications of Lemma 3.2 needed later in the analysis of the associated dividend optimization problems are now summarized in the following.

Corollary 3.3. *(A) Assume that $\eta > 0$. Then*

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}_{r+\lambda}(y) - \tilde{\psi}_{r+\lambda}(y-\eta)} \leq \frac{\psi(x)}{\psi(y) - \psi(y-\eta)} \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}_r(y) - \tilde{\psi}_r(y-\eta)}$$

for all $x \leq y$.

(B) For all $x \in I$ it holds

$$\frac{\tilde{\psi}_{r+\lambda}(x)}{\tilde{\psi}'_{r+\lambda}(x)} - x \leq \frac{\psi(x)}{\psi'(x)} - x \leq \frac{\tilde{\psi}_r(x)}{\tilde{\psi}'_r(x)} - x.$$

Proof. Noticing that

$$\frac{\psi(x)}{\psi(y) - \psi(y-\eta)} = \frac{\psi(x)/\psi(y)}{1 - \psi(y-\eta)/\psi(y)}$$

and applying the inequality (13) proves part (A). Part (B) is a direct consequence of (14). □

Before stating our main result on the general convexity properties of the increasing solution $\psi(x)$, we now present the next lemma.

Lemma 3.4. *Assume that $\phi(x) \in C^2(\mathbb{R}_+)$ is non-decreasing and that there exists $x_1 \in \mathbb{R}_+$ such that $\phi(x)$ is strictly concave on $(0, x_1)$ and strictly convex on (x_1, x_2), where $x_2 > x_1$. Define $u : (x_1, x_2) \to [0, x_1)$ via $u(z) = \inf\{y \in [0, x_1) : \phi'(y) \leq \phi'(z)\}$. Then*

$$u(z) = \begin{cases} 0, & x_2 > z > x_1,\ z \geq \tilde{\Phi}(\phi'(0)) \\ \Phi(\phi'(z)), & x_2 > z > x_1,\ z < \tilde{\Phi}(\phi'(0)), \end{cases} \tag{15}$$

where the function $\Phi : (\phi'(x_1), \phi'(x_2)) \to (0, x_1)$ is defined as $\Phi = (\phi' \mid_{(0,x_1)})^{-1}$ and $\tilde{\Phi} : (\phi'(x_1), \phi'(x_2)) \to (x_1, x_2)$ is defined as $\tilde{\Phi} = (\phi' \mid_{(x_1,x_2)})^{-1}$. Moreover, $u(z)$ is continuously differentiable for $z < \tilde{\Phi}(\phi'(0))$.

Proof. Assumptions imply that $\phi'(x)$ is a unimodal continuously differentiable function with a unique minimum at x_1. Since it is strictly decreasing on $[0, x_1)$, $\phi'(0) \leq \phi'(z)$ implies that the inequality in the definition of $u(z)$ is satisfied for all $y \in [0, x_1)$ and hence $u(z) = 0$. On the other hand, $\phi'(0) \leq \phi'(z)$ is in the present case equivalent to $z \geq \tilde{\Phi}(\phi'(0))$. For $z < \tilde{\Phi}(\phi'(0))$, the continuous differentiability of $u(z)$ on $(0, \tilde{\Phi}(\phi'(0))) \cap (x_1, x_2)$ follow from the inverse function theorem, as $\phi'(x)$ is continuously differentiable and $\phi''(x) \neq 0$ on $(0, x_1)$, implying that $\Phi(y)$ is continuously differentiable on $(\phi'(x_1), \phi'(0))$, being the inverse function of the restriction $\phi' \mid_{(0,x_1)}$. □

Given this auxiliary result, we are now in position to prove the following theorem stating a set of typically satisfied sufficient conditions under which the monotonicity properties of $\psi'(x)$ can be unambiguously characterized.

Theorem 3.5. *Assume that the net appreciation rate $\rho(x)$ satisfies the limiting inequalities $\lim_{x\to\infty} \rho(x) < 0 \leq \lim_{x\downarrow 0} \rho(x)$, that there exists a unique threshold $\hat{x} \in \mathbb{S}$ such that $\rho(x)$ is increasing on $(0, \hat{x})$ and decreasing on $(\hat{x}, \infty)$, and that $\rho(x)$ is concave on $(\hat{x}, \infty)$. Then equation $\psi''(x) = 0$ has a unique root $x^* \in (\hat{x}, \infty)$ so that $\psi''(x) \lesseqqgtr 0$ for $x \lesseqqgtr x^*$ and $x^* = \text{argmin}\{\psi'(x)\}$.*

Proof. We first establish that under our assumptions the increasing solution is locally concave on a neighborhood of the origin. To accomplish this task, we first notice that the integro-differential equation $(\mathcal{G}_r\psi)(x) = 0$ can be re-expressed as

$$I(x) = r(\psi(x) - x\psi'(x)) - \rho(x)\psi'(x) - J(x, \psi(x)), \tag{16}$$

where $I(x) = \frac{1}{2}\sigma^2(x)\psi''(x)$, and

$$J(x, \psi(x)) = \int_{(0,1)} \{\psi(x - xz) - \psi(x) + xz\psi'(x)\}\nu(dz). \tag{17}$$

Assume now that there is a set $(0, \varepsilon)$, $\varepsilon < \hat{x}$, where the increasing fundamental solution is convex. Since a convex mapping satisfying the boundary condition $\psi(0) = 0$ satisfies the inequalities $\psi'(x)x \geq \psi(x)$ and $\psi(x - xz) \geq \psi(x) - xz\psi'(x)$ for all $x \in (0, \varepsilon)$ and $z \in (0, 1)$, we find from (16) that $I(x) \leq -\rho(x)\psi'(x)$. The monotonicity of $\psi(x)$ and the positivity of $\rho(x)$ on $(0, \hat{x})$ then imply that $I(x) \leq 0$ which is a contradiction due to the assumed convexity of $\psi(x)$ on $(0, \varepsilon)$. This proves that $\psi(x)$ is locally concave on a set $(0, \varepsilon)$. We now show that $\psi(x)$

cannot become convex on $(0,\hat{x})$ and, therefore, that if equation $\psi''(x)=0$ has a root, it has to be on $(\hat{x},\infty)$. To see that this is indeed the case, we observe that if $x_1<\hat{x}$ is a root of $\psi''(x)=0$, then

$$I'(x_1)=-\rho'(x_1)\psi'(x_1)-\int_{(0,1)}\{\psi'(x_1(1-z))-\psi'(x_1)\}(1-z)\nu(dz)<0$$

due to the monotonicity of $\psi(x)$, $\psi'(x)$, and $\rho(x)$. Hence, if equation $\psi''(x)=0$ has a root, it has to be on $(\hat{x},\infty)$. In order to establish that $\psi(x)$ has to become convex at some $x_2\in(\hat{x},x_0)$, where $x_0=\rho^{-1}(0)$, assume that $\psi(x)$ is concave on the entire interval $(0,x_0)$. In that case we would have the inequalities $\psi'(x)x\leq\psi(x)$ and $\psi(x-xz)\leq\psi(x)-xz\psi'(x)$ for all $x\in(0,x_0)$ and $z\in(0,1)$. Consequently, $I(x)\geq-\rho(x)\psi'(x)$ for all $x\in(0,x_0)$. Letting $x\uparrow x_0$ then yields that $I(x_0)\geq 0$ which is a contradiction due to the assumed concavity of $\psi(x)$. Combining this observation with our previous findings shows that equation $\psi''(x)=0$ has at least one root $x^*\in(\hat{x},x_0)$.

Given these findings, our objective is now to establish that the root x^* is unique whenever $\rho(x)$ is concave on $(\hat{x},\infty)$. To observe that this is the case, we notice that (16) can be re-expressed as

$$\tilde{I}(x)=(r+\lambda)\left(\frac{\psi(x)}{S'(x)}-x\frac{\psi'(x)}{S'(x)}\right)-\tilde{\rho}(x)\frac{\psi'(x)}{S'(x)}-\tilde{J}(x), \tag{18}$$

where $\tilde{I}(x)=\frac{\sigma^2(x)\psi''(x)}{2S'(x)}$, $\tilde{\rho}(x)=\rho(x)-\lambda x(1-\bar{z})$, $S'(x)=\exp\left(-\int\frac{2\tilde{\mu}(x)dx}{\sigma^2(x)}\right)$ denotes the scale density of the associated diffusion $\tilde{X}$, and

$$\tilde{J}(x)=\int_{(0,1)}\frac{\psi(x(1-z))}{S'(x)}\nu(dz).$$

Standard differentiation yields that

$$\frac{d}{dx}\left[\frac{\psi'(x)}{S'(x)}\right]=\left((r+\lambda)\psi(x)-\int_{(0,1)}\psi(x(1-z))\nu(dz)\right)m'(x)$$

$$\frac{d}{dx}\left[\frac{\psi(x)}{S'(x)}-x\frac{\psi'(x)}{S'(x)}\right]=\left(\tilde{\rho}(x)\psi(x)+x\int_{(0,1)}\psi(x(1-z))\nu(dz)\right)m'(x)$$

and

$$\tilde{J}'(x)=\int_{(0,1)}\frac{\psi'(x(1-z))}{S'(x)}(1-z)\nu(dz)+\tilde{\mu}(x)m'(x)\int_{(0,1)}\psi(x(1-z))\nu(dz)$$

where $m'(x) = 2/(\sigma^2(x)S'(x))$ denotes the speed measure of the associated diffusion $\tilde{X}$. Hence, we find that

$$\tilde{I}'(x) = \frac{\psi'(x)}{S'(x)}\left[-\rho'(x) + \int_{(0,1)}\left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz)\right]. \qquad (19)$$

In light of the definition of $\tilde{I}(x)$ and our findings on the local concavity of $\psi(x)$ on $(0, x^*)$, it is clear that $\tilde{I}'(x^*) > 0$. Assume now that equation $\psi''(x) = 0$ has another root $y^* > x^*$ at which the increasing fundamental solution becomes locally concave again. To establish that this is impossible, we first observe that the integral term in (19) can be re-expressed as

$$\int_0^{1-\frac{u(x)}{x}}\left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz) + \\ + \int_{1-\frac{u(x)}{x}}^1\left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz),$$

where $u(x) = \inf\{y \in (0, x^*] : \psi'(y) \leq \psi'(x)\} \in C^1((x^*, y^*))$ by Lemma 3.4. It is now clear that the first term of this expression is positive due to the local convexity of $\psi(x)$ on (x^*, y^*). On the other hand, a direct application of Leibniz' rule to the second term proves

$$\frac{d}{dx}\int_{1-\frac{u(x)}{x}}^1\left(1 - \frac{\psi'(x(1-z))}{\psi'(x)}\right)(1-z)\nu(dz) = \\ \int_{1-\frac{u(x)}{x}}^1 \frac{\psi''(x)\psi'(x(1-z)) - \psi'(x)\psi''(x(1-z))(1-z)}{\psi'(x)}(1-z)\nu(dz) > 0,$$

since $\psi''(x) > 0$ on (x^*, y^*) and $\psi''(x(1-z)) < 0$ when $z > 1 - \frac{u(x)}{x}$. Combining this observation with (19), the identity $u(x^*) = x^*$, and the assumed concavity of $\rho(x)$ on $(\hat{x}, \infty)$ then proves that

$$\tilde{I}'(x) > \frac{\psi'(x)}{S'(x)}\left[-\rho'(x^*) + \int_0^1\left(1 - \frac{\psi'(x^*(1-z))}{\psi'(x^*)}\right)(1-z)\nu(dz)\right] > 0$$

for all $x \in (x^*, y^*)$. Letting $x \uparrow y^*$ now implies that $\tilde{I}'(y^*) > 0$ which is a contradiction since $\tilde{I}(x)$ should be decreasing at y^*. Hence, we find that the root x^* is unique and constitutes the global minimum of $\psi'(x)$.

□

Theorem 3.5 states a set of conditions under which $\psi'(x)$ attains a unique global minimum so that $\psi(x)$ is concave below and convex above this critical threshold. We conclude this section with the following useful observation.

Theorem 3.6. *Suppose that the assumptions of Theorem 3.5 are satisfied and define the function* $F : I \mapsto \mathbb{R}_+$ *as* $F(x) = H(x, x^*)$*. Then,*

(A) $F \in C^2(I)$, $(\mathcal{G}_r F)(x) \le 0$, $F'(x) \ge 1$, *and* $F''(x) \le 0$ *for all* $x \in I$, *and*

(B) $F(x) \ge H(x, y)$ *and* $F'(x) \ge H_x(x, y)$ *for all* $x, y \in I^2$ *and* $H_y(x, y) < 0$ *for all* $(x, y) \in \mathbb{R}_+ \times (x^*, \infty)$.

Moreover,

(C) if $\rho(x) - (1 - \bar{z})\lambda x$ *is increasing on a neighborhood of* 0 *then* $F(x) \ge \tilde{H}_{r+\lambda}(x, \tilde{x}^*(r + \lambda))$, *where* $\tilde{x}^*(r + \lambda) = \operatorname{argmin}\{\tilde{\psi}'_{r+\lambda}(x)\}$ *is the unique root of equation* $\tilde{\psi}''_{r+\lambda}(x) = 0$, *and*

(D) if $\lim_{x\to\infty}(\rho(x) + \lambda \bar{z} x) < 0$ *then* $F(x) \le \tilde{H}_r(x, \tilde{x}^*(r))$, *where* $\tilde{x}^*(r) = \operatorname{argmin}\{\tilde{\psi}'_r(x)\}$ *is the unique root of equation* $\tilde{\psi}''_r(x) = 0$.

(E) if $\rho(x) - (1 - \bar{z})\lambda x$ *is increasing on a neighborhood of* 0 *and* $\lim_{x\to\infty}(\rho(x) + \lambda \bar{z} x) < 0$ *then* $\tilde{H}_{r+\lambda}(x, \tilde{x}^*(r + \lambda)) \le H(x, x^*) \le \tilde{H}_r(x, \tilde{x}^*(r))$ *for all* $x \in I$.

Proof. (A) Clearly $F \in C^2(I)$. Since $F(x)$ is a linear function with derivative equal to 1 on $[x^*, \infty)$, it is straightforward to compute that

$$(\mathcal{G}_r F)'(x) = \rho'(x) + \int_0^1 [F'(x - xz) - 1](1 - z)\nu(dz) \tag{20}$$

for $x \ge x^*$. We have assumed that $\rho'(x)$ is negative and decreasing on $[x^*, \infty)$. Furthermore, the integral in (20) can be written as

$$\int_{1-x^*/x}^1 \left[\frac{\psi'(x - xz)}{\psi'(x^*)} - 1\right](1 - z)\nu(dz),$$

since for $x - xz \ge x^*$ the integrand vanishes. This is a decreasing function of x since $\psi(x)$ is concave on $(0, x^*)$ and $x - xz < x^*$ in the region over which we integrate here. Hence, $(\mathcal{G}_r F)'(x)$ is decreasing on (x^*, ∞) and consequently if $(\mathcal{G}_r F)'(x^*) \le 0$, then $(\mathcal{G}_r F)(x)$ is non-increasing on (x^*, ∞). But

$$(\mathcal{G}_r F)'(x^*) = \rho'(x^*) + \int_{(0,1)} \left(\frac{\psi'(x^*(1 - z))}{\psi'(x^*)} - 1\right)(1 - z)\nu(dz),$$

and this quantity was shown to be negative in the proof of Theorem 3.5. As $(\mathcal{G}_r F)(x)$ is continuous and equal to 0 for all $x < x^*$, we necessarily have

$(\mathcal{G}_r F)(x^*) = 0$. By continuity and monotonicity of $(\mathcal{G}_r F)(x)$ on (x^*, ∞), it follows that $(\mathcal{G}_r F)(x) \leq 0$ for all $x \geq x^*$. The strict concavity of $\psi(x)$ on $(0, x^*)$ then proves that $F'(x) \geq 1$ and $F''(x) \leq 0$ for all $x \in I$. Part (B) now follows directly from Theorem 3.2 in Alvarez and Virtanen (2006) since $\psi''(x) \lesseqqgtr 0$ for all $x \lesseqqgtr x^*$ and $x^* = \operatorname{argmin}\{\psi'(x)\}$. Part (C) follows from Lemma 3.2 after noticing that if $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of 0 then according to Lemma 3.1 in Alvarez and Virtanen (2006) equation $\tilde{\psi}''_{r+\lambda}(x) = 0$ has a unique root $\tilde{x}^*(r+\lambda) = \operatorname{argmin}\{\tilde{\psi}'_{r+\lambda}(x)\}$ so that $\tilde{\psi}''_{r+\lambda}(x) \lesseqqgtr 0$ for $x \lesseqqgtr \tilde{x}^*(r+\lambda)$. Establishing part (D) is entirely analogous. Part (E) finally follows from (C) and (D). □

Theorem 3.6 demonstrates that if the conditions of Theorem 3.5 are satisfied then the value $H(x, y)$ attains a unique global maximum as a function of the threshold y. Interestingly, and along the lines of the findings by Alvarez and Virtanen (2006) based on continuous diffusion processes, Theorem 3.6 proves that this maximal value $H(x, x^*)$ does not only dominate the values $H(x, y)$ for all $y \in I$, it also grows faster than any other of these values. Theorem 3.6 also establishes a set of sufficient conditions under which the two associated values $\tilde{H}_{r+\lambda}(x, y)$ and $\tilde{H}_r(x, y)$ attain a unique global maximum as functions of the arbitrary threshold y. Whenever these optimal thresholds exist the value $H(x, x^*)$ belongs into the region bounded by the resulting values.

4 Optimal Singular Control of Dividends

We are now in a position to state our main result, which characterizes the optimal singular controls for the considered class of jump diffusions and echoes the similar result obtained in Alvarez and Virtanen (2006) (Lemma 3.1) for continuous diffusions.

Theorem 4.1. *Assume that the assumptions of Theorem 3.5 are satisfied. Then the value of the singular control problem is given by $V_S(x) = H(x, x^*)$. The value is twice continuously differentiable, monotonically increasing and concave. Moreover, the marginal value (* Tobin's marginal q *) of the singular control reads* *as*

$$V_S'(x) = \psi'(x) \sup_{y \geq x} \left\{ \frac{1}{\psi'(y)} \right\} = \begin{cases} 1 & x \geq x^* \\ \frac{\psi'(x)}{\psi'(x^*)} & x < x^*. \end{cases} \tag{21}$$

The corresponding optimal singular control consists of an initial impulse (lump sum dividend) $\xi_{0-} = (x - x^*)^+$ *and a barrier strategy where all retained earnings in excess of* x^* *are instantaneously paid out as dividends.*

Proof. For notational convenience, we shall denote the proposed value function as $\tilde{V}(x)$ and the value function of the singular control problem as $V(x)$. Let $D \in \mathcal{A}$ be an arbitrary admissible policy and denote $J^D(x) = \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s$. Applying the generalized Itô formula (see Protter (2004) Theorem II.32) to the mapping $(t, x) \mapsto e^{-rt}\tilde{V}(X_t^D)$ yields

$$\begin{aligned} \mathbb{E}_x \left[e^{-r\tau_N} \tilde{V}(X_{\tau_N}^D) \right] &= \tilde{V}(x) + \mathbb{E}_x \int_{0+}^{\tau_N} e^{-rs} (\mathcal{G}_r \tilde{V})(X_s^D) ds \\ &+ \mathbb{E}_x \int_{0+}^{\tau_N} e^{-rs} \{ \tilde{V}(X_{s-}^D + (\Delta D)_s) - \tilde{V}(X_{s-}^D) \} ds \\ &- \mathbb{E}_x \int_{0+}^{\tau_N} e^{-rs} \tilde{V}'(X_{s-}^D) dD_s, \end{aligned} \tag{22}$$

where $\tau_N = N \wedge \tau_0^D \wedge \inf\{t \geq 0 : X_t^D \geq N\}$ is an increasing sequence of almost surely finite stopping times converging to τ_0^D as $N \to \infty$. It is now clear from our Theorem 3.6 that the proposed value function is nonnegative and twice continuously differentiable and that it satisfies the inequalities $\tilde{V}'(x) \geq 1$ and $(\mathcal{G}_r F)(x) \leq 0$ for all $x \in I$. Combining these observations with (22) implies

$$0 \leq \mathbb{E}_x \left[e^{-r\tau_N} \tilde{V}(X_{\tau_N}^D) \right] \leq \tilde{V}(x) - \mathbb{E}_x \int_{0+}^{\tau_N} e^{-rs} dD_s.$$

This inequality and the monotone convergence theorem then imply that

$$\tilde{V}(x) \geq \mathbb{E}_x \int_{0+}^{\tau_N} e^{-rs} dD_s \to \mathbb{E}_x \int_{0+}^{\tau_0^D} e^{-rs} dD_s$$

as $N \to \infty$. Thus $\tilde{V}(x) \geq J^D(x)$ for any $D \in \mathcal{A}$ and so $\tilde{V}(x) \geq V(x)$.

Denote now the proposed dividend strategy described in the theorem by $\hat{D}$. Under the proposed policy we have $X_t^D \in (0, x^*]$ t-almost everywhere, implying thus that$(\mathcal{G}_r V)(X_t^D) = 0$ t-almost everywhere. Hence, (22) takes now the form

$$\mathbb{E}_x \left[e^{-r\tau_N} \tilde{V}(X_{\tau_N}^{\hat{D}}) \right] = \tilde{V}(x) - \mathbb{E}_x \int_{0+}^{\tau_N} e^{-rs} \tilde{V}'(X_{s-}^{\hat{D}}) d\hat{D}_s. \tag{23}$$

However, since the proposed dividend policy increases only when the underlying process hits the threshold x^* and, therefore, when $V'(X_s) = 1$ we find that (23) can be re-expressed as

$$\tilde{V}(x) = \mathbb{E}_x \left[e^{-r\tau_N} \tilde{V}(X^{\hat{D}}_{\tau_N}) \right] + \mathbb{E}_x \int_0^{\tau_N} e^{-rs} d\hat{D}_s.$$

Recalling that either $\tau_N \to \infty$ or $X^{\hat{D}}_{\tau_N} = 0$ for N large enough, letting $N \to \infty$ then gives $\tilde{V}(x) = J^{\hat{D}}(x)$ and consequently $\tilde{V}(x) \leq V(x)$. But then $\tilde{V}(x) = V(x)$.

□

The capital theoretic implications of Theorem 4.1 are in line with the ones stated in Alvarez and Virtanen (2006): firstly, the optimal dividend threshold is attained on the set where net appreciation rate $\rho(x)$ of the underlying reserve is positive and thus dividends are paid out on the set where the expected per capita rate at which the reserves are increasing dominate the opportunity cost of investment; secondly, since the optimal dividend threshold is attained on the set where the net appreciation rate of the underlying reserve is decreasing, at the optimum the marginal yield accrued from retaining yet another marginal unit of stock undistributed is smaller than the interest rate r. Thus, the optimal dividend policy diverges from the deterministic golden rule of capital accumulation in the present jump-diffusion case as well. An important implication of Theorem 4.1 and Theorem 3.6 is now summarized in the following.

Corollary 4.2. *Assume that the assumptions of Theorem 3.5 are satisfied, that* $\rho(x) - (1 - \bar{z})\lambda x$ *is increasing on a neighborhood of* 0 *and that* $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$. *Then, for all* $x \in I$ *it holds*

$$\tilde{V}^{r+\lambda}_S(x) \leq V_S(x) \leq \tilde{V}^r_S(x),$$

where

$$\tilde{V}^\theta_S(x) = \sup_{D\in\mathcal{A}} \mathbb{E}_x \int_0^\infty e^{-\theta s} d\tilde{D}_s,$$

and

$$d\tilde{X}^{\tilde{D}}_t = \tilde{\mu}(\tilde{X}^{\tilde{D}}_t)dt - \sigma(\tilde{X}^{\tilde{D}}_t)dW_t - d\tilde{D}_t, \tilde{X}^{\tilde{D}}_0 = x.$$

Proof. Since $V_S(x) = H(x, x^*)$ and $\tilde{V}_S^{\theta}(x) = \tilde{H}_\theta(x, \tilde{x}^*(\theta))$ the result follows from Theorem 3.6. $\square$

Corollary 4.2 states a set of conditions under which the value of the optimal singular dividend policy is bounded by the value of an associated singular stochastic control problem of the continuous diffusion $\tilde{X}$. It is worth noticing that even though the jump intensity λ does not affect the existence of the optimal threshold x^* it affects the existence of an optimal policy for the associated problems. As the jump intensity λ increases the local growth rate of $\rho(x) - (1-\bar{z})\lambda x$ decreases and eventually vanishes (provided that $\mu'(0+) < \infty$). At the critical level $(1-\bar{z})\lambda = \mu'(0+)$ the optimal policy associated to the smallest value becomes trivial (instantaneous liquidation) and $\tilde{V}_S^{\lambda+r}(x) = x$. Analogous conclusions can be naturally drawn for the highest value as well.

Having analyzed the considered singular control problem, we now proceed in our analysis and study the associated optimal liquidation problem. An important implication of Theorem 3.6 and Theorem 4.1 characterizing the relationship between the optimal singular dividend policy and the value of the optimal liquidation policy is now summarized in the following representation theorem for the associated optimal stopping problem.

Theorem 4.3. *Suppose assumptions of Theorem 3.5 are satisfied. Then the value of the associated optimal stopping problem* (9) *reads as*

$$V_{OSP}(x) = \psi(x) \sup_{y \geq x} \left\{ \frac{y}{\psi(y)} \right\} = H(x, x_0^*), \tag{24}$$

where the optimal stopping boundary $x_0^* \geq x^*$ is the unique root of $\psi(x) = x\psi'(x)$. Moreover, $V_{OSP}(x) \leq V_S(x)$ and $V_{OSP}'(x) \leq V_S'(x)$ for all $x \in I$.

Proof. First we need to establish existence and uniqueness of x_0^*. For this, note that by Theorem 3.5, under our assumptions $\psi(x)$ is strictly concave for $x < x^*$ and strictly convex for $x > x^*$. Hence,

$$D_x \left[\frac{x}{\psi(x)} \right] = \frac{\psi(x) - x\psi'(x)}{\psi^2(x)} > 0 \tag{25}$$

for all $x < x^*$ and $\lim_{x\to\infty} x/\psi(x) = 0$. This implies the existence of $x_0^* \geq x^*$ such that $\psi(x_0^*) = x_0^* \psi'(x_0^*)$. Moreover, the convexity of $\psi(x)$ on (x^*, ∞) implies that

$$D_x\left[\psi(x) - x\psi'(x)\right] = -x\psi''(x) < 0$$

for all $x > x_0^*$. Thus $x/\psi(x)$ is decreasing for all $x > x_0^*$ and so x_0^* is unique.

We will use the notation $v(x) = \psi(x)\sup_{y\geq x}\{y/\psi(y)\}$. It is immediate from the definition that $v(x) \geq x$ for all $x \in I$, that $v \in C^1(I)\cap C^2(I\backslash\{x_0^*\})$, and that $|v''(x_0^*\pm)| < \infty$. We will now prove that $v(x)$ is r-superharmonic with respect to X. It is clear that since $v(x) = \psi(x)/\psi'(x_0^*)$ on $(0, x_0^*)$ and $(\mathcal{G}_r\psi)(x) = 0$, we have $(\mathcal{G}_r v)(x) = 0$ for all $x \in (0, x_0^*)$. To establish the r-superharmonicity of $v(x)$ on $[x_0^*, \infty)$ we first note that since $\psi(x_0^*) = \psi'(x_0^*)x_0^*$ and $\psi''(x_0^*) > 0$ the identity $(\mathcal{G}_r\psi)(x) = 0$ implies that

$$0 > -\frac{1}{2}\sigma^2(x_0^*)\psi''(x_0^*) = \rho(x_0^*)\psi'(x_0^*) + \int_0^1 \left(\psi(x_0^*(1-z)) - (1-z)x_0^*\psi'(x_0^*)\right)\nu(dz).$$

Dividing this inequality with $\psi'(x_0^*)$ then shows that

$$\lim_{x\to x_0^*+}(\mathcal{G}_r v)(x) = \rho(x_0^*) + \int_0^1 \left(\frac{\psi(x_0^*(1-z))}{\psi'(x_0^*)} - (1-z)x_0^*\right)\nu(dz) < 0.$$

We will now prove that $(\mathcal{G}_r v)(x)$ is non-increasing on (x_0^*, ∞) and, therefore, that $(\mathcal{G}_r v)(x) < 0$ for all $x \in (x_0^*, \infty)$. Differentiating the functional $(\mathcal{G}_r v)(x)$ and applying the inequalities $(\mathcal{G}_r V_s)'(x) \leq 0$ and $v'(x) \leq V_S'(x)$ established in Theorem 3.6 demonstrates that for all $x \in (x_0^*, \infty)$ we have

$$\begin{aligned}(\mathcal{G}_r v)'(x) &= \rho'(x) + \int_{(0,1)} (v'(x(1-z)) - 1)(1-z)\nu(dz) \\ &\leq (\mathcal{G}_r V_s)'(x) + \int_{1-x_0^*/x}^{1-x^*/x} (v'(x(1-z)) - 1)(1-z)\nu(dz) \leq 0\end{aligned}$$

since $\psi(x)$ is convex on (x^*, x_0^*). Hence $(\mathcal{G}_r v)(x) \leq 0$ for all $x \in I$. Consequently, $v(x)$ constitutes a nonnegative r-superharmonic majorant of x and, therefore, $v(x) \geq V_{\mathrm{OSP}}(x)$, as the latter is by the general theory the *least* r-superharmonic majorant of x.

In order to establish the opposite inequality we first observe that for $y > x$

$$\mathbb{E}_x\left[e^{-r\tau_y}X_{\tau_y}\right] = y\mathbb{E}_x\left[e^{-r\tau_y}\right] = y\frac{\psi(x)}{\psi(y)}$$

and for $y \leq x$, $\mathbb{E}_x\left[e^{-r\tau_y}X_{\tau_y}\right] = x$. Hence the choice $y = \mathrm{argmax}[z/\psi(z)] = x_0^*$ yields

$$v(x) = \mathbb{E}_x\left[e^{-r\tau_{x_0^*}}X_{\tau_{x_0^*}}\right] \leq V_{\mathrm{OSP}}(x).$$

Thus $V_{\mathrm{OSP}}(x) \leq v(x) \leq V_{\mathrm{OSP}}(x)$, and the claimed representation is proved. The last two claims follow in a straightforward fashion from Theorem 3.6, since $V_{\mathrm{OSP}}(x) = H(x, x_0^*)$. □

We wish to point out that the representation of the value of the stopping problem given in the previous theorem holds also for more general jump diffusions and reward functions under some additional conditions, as has been shown in Alvarez and Rakkolainen (2006). An interesting implication of Theorem 4.3 and Lemma 3.2 extending the observation of Corollary 4.2 to the optimal liquidation case as well is now summarized in the following.

Corollary 4.4. *Assume that the assumptions of Theorem 3.5 are satisfied, that* $\rho(x) - (1-\bar{z})\lambda x$ *is increasing on a neighborhood of* 0 *and that* $\lim_{x\to\infty}(\rho(x) + \lambda \bar{z} x) < 0$. *Then, for all* $x \in I$ *it holds*

$$\tilde{V}_{OSP}^{r+\lambda}(x) \leq V_{OSP}(x) \leq \tilde{V}_{OSP}^{r}(x), \tag{26}$$

where

$$\tilde{V}_{OSP}^{\theta}(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-\theta\tau} \tilde{X}_\tau \right] = \tilde{\psi}_\theta(x) \sup_{y \geq x} \left[\frac{y}{\tilde{\psi}_\theta(y)} \right].$$

Moreover, $\tilde{x}_0^*(r+\lambda) < x_0^* < \tilde{x}_0^*(r)$, *where* $\tilde{x}_0^*(\theta)$ *denotes the unique root of the optimality condition* $\tilde{\psi}_\theta(\tilde{x}_0^*(\theta)) = \tilde{\psi}_\theta'(\tilde{x}_0^*(\theta))\tilde{x}_0^*(\theta)$.

Proof. Since $V_{OSP}(x) = H(x, x_0^*)$ and $\tilde{V}_{OSP}^{\theta}(x) = \tilde{H}_\theta(x, \tilde{x}^*(\theta))$ inequality (26) follows from Theorem 3.6 and Lemma 3.2. The ordering $\tilde{x}_0^*(r+\lambda) < x_0^* < \tilde{x}_0^*(r)$ is a direct implication of Corollary 3.3, the proof of Theorem 4.3, and the inequality $\tilde{\psi}_\theta''(x) \lesseqgtr 0$ for $x \lesseqgtr \tilde{x}_0^*(\theta)$. □

5 Optimal Impulse Control of Dividends

Let us next consider the problem of determining the *optimal impulse control* in our Lévy diffusion model in case where each dividend distribution incurs a fixed cost $c > 0$. An impulse type dividend control consists of an increasing sequence of $\mathbb{F}$-stopping times $\tau = (\tau(i))$, $i \leq N \leq \infty$, (intervention times) and a corresponding sequence of non-negative impulses $\xi = (\xi(i))$, $i \leq N \leq \infty$, (interventions). The standard approach is to seek an optimal impulse control $\hat{\nu} = (\hat{\tau}, \hat{\xi})$ in the whole class of admissible impulse controls

$$\mathcal{V} = \left\{(\tau, \xi) \,:\, \tau(i) \in \mathcal{T},\, 0 \le \xi(i) \le X_{\tau(i)},\, 1 \le i \le N\right\}$$

such that the expected cumulative present value of the policy,

$$J^{\tau,\xi}(x) = \mathbb{E}_x \left[\sum_{i=1}^{N} e^{-r\tau(i)} (\xi(i) - c) \right],$$

is maximized, that is, $(\hat{\tau}, \hat{\xi})$ should satisfy

$$V_{\mathrm{I}}^c(x) = \sup_{(\tau,\xi)\in\mathcal{V}} J^{\tau,\xi}(x) = \mathbb{E}_x \left[\sum_{i=1}^{N} e^{-r\hat{\tau}(i)} (\hat{\xi}(i) - c) \right].$$

The associated optimal stopping problem is defined as

$$V_{\mathrm{OSP}}^c(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x \left[e^{-r\tau}(X_\tau - c) \right]. \tag{27}$$

The following analogue of Theorem 4.3 and its Corollary 4.4 holds for this stopping problem.

Lemma 5.1. *(A) Suppose assumptions of Theorem 3.5 are satisfied. Then the value of the associated optimal stopping problem* (27) *reads as*

$$V_{OSP}^c(x) = \psi(x) \sup_{y\ge x} \left\{ \frac{y-c}{\psi(y)} \right\} = H(x, x_c^*) \tag{28}$$

where the optimal stopping boundary $x_c^* \ge x_0^* \ge x^*$ is the unique root of $\psi(x) = (x-c)\psi'(x)$. Moreover, $V_{OSP}^c(x) \le V_S(x)$ and $V_{OSP}^{c\,\prime}(x) \le V_S'(x)$ for all $x \in I$.

(B) If also $\rho(x) - (1-\bar{z})\lambda x$ is increasing on a neighborhood of 0 *and* $\lim_{x\to\infty}(\rho(x) + \lambda\bar{z}x) < 0$ *then, for all $x \in I$ it holds*

$$\tilde{V}_{OSP}^{r+\lambda,c}(x) \le V_{OSP}^c(x) \le \tilde{V}_{OSP}^{r,c}(x), \tag{29}$$

where

$$\tilde{V}_{OSP}^{\theta,c}(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-\theta\tau}(\tilde{X}_\tau - c) \right] = \tilde{\psi}_\theta(x) \sup_{y\ge x} \left[\frac{(y-c)}{\tilde{\psi}_\theta(y)} \right].$$

Moreover, $\tilde{x}_c^*(r+\lambda) < x_c^* < \tilde{x}_c^*(r)$, where $\tilde{x}_c^*(\theta)$ denotes the unique root of the optimality condition $\tilde{\psi}_\theta(\tilde{x}_c^*(\theta)) = \tilde{\psi}_\theta'(\tilde{x}_c^*(\theta))\tilde{x}_c^*(\theta)$.

Proof. This is simply a straightforward replication of the proof of Theorem 4.3 and its Corollary 4.4, *mutatis mutandis.* □

We shall follow an approach similar to the one adopted in Alvarez and Virtanen (2006) and determine the optimal choice within a more restricted class of impulse dividend controls based on a single threshold level and a fixed dividend size. We will then proceed to give reasonably general conditions under which this optimal choice in the restricted class is, in fact, optimal also in the larger class $\mathcal{V}$. To avoid unnecessary duplication, we will mostly refer to Alvarez and Virtanen (2006) for detailed arguments when the presence of jumps does not affect the analysis.

Consider a dividend policy (τ_y, η) such that a constant dividend η is paid out when the underlying reaches a specified threshold level y, and in case $x > y$ an exceptional, state-dependent initial dividend $x - y + \eta$ is paid out to bring the state below level y. By relying on a similar reasoning as in Section 4 of Alvarez and Virtanen (2006) we find that the value of such a policy has the following representation in terms of the minimal r-excessive map $\psi(x)$:

$$J^{\tau_y,\eta}(x) = F_c(x) = \begin{cases} x - y + \frac{(\eta-c)\psi(y)}{\psi(y)-\psi(y-\eta)}, & x \geq y \\ \frac{(\eta-c)\psi(x)}{\psi(y)-\psi(y-\eta)}, & x < y. \end{cases} \tag{30}$$

Consider now the inequality constrained nonlinear programming problem

$$\sup_{\eta\in[0,y],\, y\in I} h(\eta, y) = \sup_{\eta\in[0,y],\, y\in I} \frac{(\eta - c)}{\psi(y) - \psi(y - \eta)}. \tag{31}$$

If there exists a unique optimal pair (η_c^*, y_c^*) maximizing $h(\eta, y)$, we can define

$$F_c^*(x) = \begin{cases} x - y_c^* + h(\eta_c^*, y_c^*)\psi(y_c^*), & x \geq y_c^* \\ h(\eta_c^*, y_c^*)\psi(x), & x < y_c^*. \end{cases} \tag{32}$$

It is an immediate consequence of the necessary first order conditions for optimality in (31) that

$$F_c^*(x) = H(x, y_c^*) = \begin{cases} x - y_c^* + \frac{\psi(y_c^*)}{\psi'(y_c^*)}, & x \geq y_c^* \\ \frac{\psi(x)}{\psi'(y_c^*)}, & x < y_c^* \end{cases} \tag{33}$$

(see Alvarez and Virtanen (2006)). It is now clear that $F_c^*(x)$ belongs to the class of mappings considered in Theorem 3.6 and hence $F_c^*(x) \leq V_S(x)$ and $F_c^{*\prime}(x) \leq V_S'(x)$ (given existence and uniqueness of (η_c^*, y_c^*)). We will now establish a set of sufficient conditions for the existence of a unique optimal pair (η_c^*, y_c^*) solving (31).

Lemma 5.2. *Suppose, in addition to the assumptions of Theorem 3.5, that* $\lim_{x\downarrow 0} \psi'(x) = \infty$. *Then there exists a unique pair* $(\eta_c^*, y_c^*) \in (c, y_c^*) \times (x^*, x_c^*)$, *which satisfies the necessary first order conditions* $\psi'(y_c^*) = \psi'(y_c^* - \eta_c^*)$ *and* $\psi(y_c^*) - \psi(y_c^* - \eta_c^*) = \psi'(y_c^* - \eta_c^*)(\eta_c^* - c)$.

Proof. Under assumptions of Theorem 3.5 there exists a unique x^* such that $\psi'(x)$ is strictly decreasing (increasing) on $(0, x^*)$ $((x^*, \infty))$. By strict convexity of $\psi(x)$, $\lim_{x\to\infty} \psi'(x) = \infty$. If $\lim_{x\downarrow 0} \psi'(x) = \infty$, then this implies that for any $y \in (x^*, \infty)$ there exists a unique $\hat{y} \in (0, x^*)$ such that $\psi'(y) = \psi'(\hat{y})$. Moreover, we can define $\hat{x}^* = x^*$. Hence the function $y \mapsto \hat{y}$ from $[x^*, \infty)$ onto $(0, x^*]$ is well-defined. It is decreasing and continuous (even C^1, see the proof of Lemma 3.4). Consider then the continuous function

$$L(y) = \psi(y) - \psi(\hat{y}) - \psi'(\hat{y})(y - \hat{y}) + c \cdot \psi'(\hat{y})$$

defined for $y \in [x^*, \infty)$. Now $L(x^*) = c \cdot \psi'(x^*) > 0$ and

$$\begin{aligned} L(x_c^*) &= \psi(x_c^*) - \psi(\hat{x}_c^*) - \psi'(\hat{x}_c^*)(x_c^* - \hat{x}_c^*) + c \cdot \psi'(\hat{x}_c^*) \\ &= (\psi(x_c^*) - \psi'(\hat{x}_c^*)x_c^*) - \psi(\hat{x}_c^*) + \psi'(\hat{x}_c^*)\hat{x}_c^* + c \cdot \psi'(\hat{x}_c^*) \\ &= (\psi(x_c^*) - \psi'(x_c^*)x_c^*) - \psi(\hat{x}_c^*) + \psi'(\hat{x}_c^*)\hat{x}_c^* + c \cdot \psi'(\hat{x}_c^*) \\ &= -c \cdot \psi'(x_c^*) - (\psi(\hat{x}_c^*) - \psi'(\hat{x}_c^*)\hat{x}_c^*) + c \cdot \psi'(\hat{x}_c^*) < 0, \end{aligned}$$

since $\psi(x) - \psi'(x)x > 0$ for all $x < x^*$. Thus there exists $y_c^* \in (x^*, x_c^*)$ such that $L(y_c^*) = 0$, in other words, the choice $(\eta, y) = (y_c^* - \hat{y}_c^*, y_c^*)$ satisfies the first order conditions for optimality in (31). To establish uniqueness of this solution, note that

$$L'(y) = \psi''(\hat{y}) \cdot \hat{y}'(y) \cdot (c - y + \hat{y}),$$

whose sign is determined by the last factor on the right hand side, both other factors being always negative for $y \in (x^*, \infty)$. As $c - y + \hat{y} = c > 0$ for $y = x^*$,

$\lim_{y\to\infty}(c-y+\hat{y})=-\infty$ and furthermore $c-y+\hat{y}$ is decreasing in y, we see that $L(y)$ is a unimodal function with a unique maximum. Since $L(x^*)=c>0$, the root of $L(y)=0$ is necessarily unique. □

Having established sufficient conditions for existence and uniqueness of the pair (η_c^*, y_c^*) satisfying the necessary optimality conditions of (31), we now proceed to state our second main theorem, whose proof requires a verification lemma.

Lemma 5.3. *Assume that the mapping $g: I \to I$ is strictly increasing and satisfies the conditions $g \in C^1(I)\cap C^2(I\setminus\mathcal{D})$, where $\mathcal{D}$ is a set of zero measure, and $|g''(x\pm)|<\infty$ for all $x\in\mathcal{D}$. Suppose further that g satisfies the quasi-variational inequality*

$$\sup_{\eta\in[0,x]}\left\{\eta-c+g(x-\eta)\right\}\leq g(x)$$

for all $x\in I$ and the variational inequality $\left(\mathcal{G}_r g\right)(x)\leq 0$ for all $x\in I\setminus\mathcal{D}$. Then $g(x)\geq V_I^c(x)$ for all $x\in I$.

Proof. The estimations in the proof of Lemma 2.1 in Alvarez and Virtanen (2006) go through also in our setting, when one notices two things. Firstly, an application of the dominated convergence theorem yields

$$\int_0^1 \psi_k(x-xz)\mathfrak{m}(dz)\to\int_0^1\psi(x-xz)\mathfrak{m}(dz)$$

as $k\to\infty$, for any $x\in I$, and thus uniformly on compact subsets of I. Hence the approximation result from Appendix D in Øksendal (2003) used in Alvarez and Virtanen (2006) is valid also in our jump diffusion model. Secondly, for a spectrally negative jump diffusion X and an increasing non-negative function g

$$\begin{aligned} g(X^\nu_{\tau_j-})-g(X^\nu_{\tau_j}) &= g(X^\nu_{\tau_j-})-g(X^\nu_{\tau_j-}-\eta_{\tau_j}-|\Delta X^\nu_{\tau_j}|)\\ &\geq g(X^\nu_{\tau_j-})-g(X^\nu_{\tau_j-}-\eta_{\tau j}), \end{aligned}$$

which ensures that the inequalities derived in Alvarez and Virtanen (2006) remain valid in our model. □

Theorem 5.4. *Suppose the assumptions of Lemma 5.2 are satisfied. Then*

$$V_I^c(x)=J^{\tau_{y_c^*},\eta_c^*}(x)=F_c^*(x)=H(x,y_c^*),$$

where $y_c^* \in (x^*, x_c^*)$ and $\eta_c^* = y_c^* - \hat{y}_c^*$ solve (31). *In other words, the value of the optimal single threshold dividend policy coincides with the value of the optimal impulse control problem.*

Proof. As the single threshold dividend policy $\nu = \nu(\eta_c^*, y_c^*)$ is clearly an admissible impulse control, we have $F_c^*(x) \leq V_{\mathrm{I}}^c(x)$. To establish the converse inequality, by Lemma 5.3 it is enough to show that the increasing function $F_c^*(x)$ is sufficiently smooth and satisfies the relevant quasi-variational inequalities. It is easy to see by standard differentiations that $F_c^*(x) \in C^1(I) \cap C^2(I \setminus \{y_c^*\})$ and that $\lim_{x \downarrow y_c^*} |F_c^{*\prime\prime}(x)| = 0$ and $\lim_{x \uparrow y_c^*} |F_c^{*\prime\prime}(x)| < \infty$. By boundedness on compacts of continuous maps and the fact that $X_t^\nu \leq y_c^*$, t-almost everywhere, we furthermore have $\lim_{t \to \infty} \mathbb{E}_x \left[e^{-rt} F_c^*(X_t^\nu)\right] = 0$ for all $x \in I$. To see that $F_c^*(x)$ satisfies the variational inequality, note that for $x < y_c^*$

$$\left(\mathcal{G}_r F_c^*\right)(x) = \left(\psi'(y_c^*)\right)^{-1} \left(\mathcal{G}_r \psi\right)(x) = 0,$$

and for $x \geq y_c^*$, by Theorem 3.6,

$$\begin{aligned} \left(\mathcal{G}_r F_c^*\right)'(x) &= \rho'(x) + \int_0^1 \left\{F_c^{*\prime}(x - xz) - 1\right\}(1 - z)\nu(dz) \\ &\leq (\mathcal{G}_r V_s)'(x) + \int_{1 - y_c^*/x}^{1 - x^*/x} (F_c^{*\prime}(x(1 - z)) - 1)(1 - z)\nu(dz) \leq 0, \end{aligned}$$

since $\psi(x)$ is convex on (x^*, y_c^*). This implies that $\left(\mathcal{G}_r F_c^*\right)(x)$ is decreasing for $x \geq y_c^*$ and hence $\left(\mathcal{G}_r F_c^*\right)(x) \leq 0$ for all $x \in I$. Finally, to establish that the quasi-variational inequality $F_c^*(x) \geq \sup_{\eta \in [0,x]} \left[\eta - c + F_c^*(x - \eta)\right]$ holds, we may proceed exactly as in Appendix E in Alvarez and Virtanen (2006). Thus, since $F_c^*(x)$ satisfies the quasi-variational inequalities, by Lemma 5.3, $F_c^*(x) \geq V_{\mathrm{I}}^c(x)$ and hence $F_c^*(x) = V_{\mathrm{I}}^c(x)$. □

Results obtained in this section are similar to the ones obtained for continuous linear diffusions in Alvarez and Virtanen (2006) and highlight the similarities in behavior of continuous diffusions and spectrally negative jump diffusions with natural boundaries and geometric jumps. Along the lines of our previous analysis, we are now in position to establish the following interesting comparison result extending our sandwiching results to the present setting as well.

Theorem 5.5. *For all* $\eta \in [c, y]$ *and* $x \in I$ *we have* $\tilde{K}_{r+\lambda}(x) \leq F_c(x) \leq \tilde{K}_r(x)$, *where the function* $\tilde{K}_\theta : I \mapsto \mathbb{R}_+$ *is defined as*

$$\tilde{K}_\theta(x) = \begin{cases} x - y + \frac{(\eta - c)\tilde{\psi}_\theta(y)}{\tilde{\psi}_\theta(y) - \tilde{\psi}_\theta(y-\eta)}, & x \geq y \\ \frac{(\eta - c)\tilde{\psi}_\theta(x)}{\tilde{\psi}_\theta(y) - \tilde{\psi}_\theta(y-\eta)}, & x < y. \end{cases} \tag{34}$$

Consequently, if the conditions of Lemma 5.2 are satisfied, $\rho(x) - (1-\bar{z})\lambda x$ *is increasing on a neighborhood of* 0, $\lim_{x\to\infty}(\rho(x) + \lambda \bar{z} x) < 0$, *and* $\lim_{x\downarrow 0} \tilde{\psi}'_{r+\lambda}(x) = \lim_{x\downarrow 0} \tilde{\psi}'_r(x) = \infty$, *then* $\tilde{H}_{r+\lambda}(x, \tilde{y}^*_c(r+\lambda)) \leq V^c_I(x) \leq \tilde{H}_r(x, \tilde{y}^*_c(r))$, *where* $(\tilde{y}^*_c(\theta), \tilde{\eta}^*_c(\theta))$ *denotes the unique pair maximizing the function*

$$\frac{(\eta - c)}{\tilde{\psi}_\theta(y) - \tilde{\psi}_\theta(y - \eta)}.$$

Proof. The inequality $\tilde{K}_{r+\lambda}(x) \leq F_c(x) \leq \tilde{K}_r(x)$ is a direct consequence of part (A) of our Corollary 3.3. As was established in Lemma 4.1 of Alvarez and Virtanen (2006), our assumptions guarantee the existence and uniqueness of the optimal pairs $(\tilde{y}^*_c(r+\lambda), \tilde{\eta}^*_c(r+\lambda))$ and $(\tilde{y}^*_c(r), \tilde{\eta}^*_c(r))$. Combining this observation with the result of Theorem 5.4 completes our proof. □

6 Explicit Illustration: Logistic Jump Diffusion

To illustrate our general results with a particular example, we consider a *logistic jump diffusion* given by

$$dX_t = X_t \left\{ a(b - X_t)dt + \sigma dW_t - \int_0^1 zN(dt, dz) \right\}, \tag{35}$$

where parameters $a > 0$, $b > 0$, $\sigma > 0$ and the associated Lévy measure of the compensated compound Poisson process N is $\nu = \lambda \mathfrak{m}(dz)$, where $\mathfrak{m}$ is the relative jump size distribution defined on $(0, 1)$. The downside risk is thus characterized by the jump intensity λ and the form of the jump size distribution. If $ab > r$, the considered jump diffusion satisfies the conditions of Theorem 4.1. In this chapter we will take the relative jump size to be Beta(α, β) distributed. This allows us to consider both symmetric and skewed distributions by varying the parameters α and β. We assume that the discount rate $r = 0.025$ and the fixed transaction cost $c = 0.05$.

With regard to analyzing the effect of λ, we wish to point out that care may be needed when dealing with large values of λ, since in the limit $\lambda \to \infty$ we get a spectrally negative compound Poisson process with drift $\lambda \overline{z} > 0$, which (being a martingale) does *not* satisfy the classical Cramer–Lundberg net profit condition and hence oscillates and will hit 0 in finite time almost surely, violating our assumptions on the boundary behavior of the jump diffusion.

We are interested in the effect of introducing jumps – downside risk – on the optimal thresholds. The benchmark case is now the absence of downside risk, $\lambda = 0$, in which case the associated integro-differential equation $\mathcal{G}_r u = 0$ reduces to a linear ordinary second order differential equation, whose increasing fundamental solution $\psi(x)$ can be expressed in terms of the Kummer confluent hypergeometric function. Optimal boundaries for the singular control, impulse control and stopping problems (respectively) can then be solved from equations

$$\begin{aligned} &\psi^{''}(x) = 0, \\ &\begin{cases} \psi(y) - \psi(y-\eta) = \psi^{'}(y-\eta)(\eta - c) \\ \psi^{'}(y) = \psi^{'}(y-\eta) \text{ and} \end{cases} \\ &\psi(x) = \psi^{'}(x)(x-c). \end{aligned} \tag{36}$$

This yields with the assumed parameter values the first row of Table 1. For nonzero intensities λ, the integro-differential equation is not (semi-)explicitly solvable except in the case $\alpha = \beta = 1$, i.e. when relative jump sizes are uniformly distributed. In this special case the integro-differential equation can be reduced to a third order linear differential equation by considering $\Phi(x) := \int_0^x \psi(y)dy$; the obtained differential equation is then solvable in terms of generalized hypergeometric functions. For the general case, $\lambda \neq 0$ and either $\alpha \neq \beta$ or $\alpha \neq 1$, we can obtain a sufficiently good approximation for $\psi(x)$ by applying the Frobenius method. That is, we will assume that the solution ψ of $\mathcal{G}_r u = 0$ is of form $\psi(x) = x^{\varsigma} \sum_{n=0}^{\infty} \gamma_n x^n$, plug this into $\mathcal{G}_r u = 0$ and solve the resulting indicial (integral) equation

$$\left(ab + \lambda \frac{\alpha}{\alpha + \beta}\right)\varsigma + \frac{1}{2}\sigma^2 \varsigma(\varsigma - 1) - \tilde{r} + \frac{\lambda}{B(\alpha, \beta)} \int_0^1 (1-z)^{\varsigma} z^{\alpha - 1}(1-z)^{\beta - 1} dz = 0$$

for ς and the recursion relation

$$\left(ab+\lambda\frac{\alpha}{\alpha+\beta}\right)\gamma_n(\varsigma+n)-a\gamma_{n-1}(\varsigma+n-1)+\frac{\sigma^2}{2}\gamma_n(\varsigma+n)(\varsigma+n-1)-$$
$$-\left(\tilde{r}-\frac{\lambda}{B(\alpha,\beta)}\int_0^1(1-z)^{\varsigma+n}z^{\alpha-1}(1-z)^{\beta-1}dz\right)\gamma_n=0.$$

for $\{\gamma_n\}$. If $\varsigma>0$ solves the indicial equation and the obtained sequence of coefficients $\{\gamma_n\}$ converges to zero fast enough, a sufficiently good numerical approximation for ψ is obtained by truncating the infinite series in $\psi(x)=x^\varsigma\sum_{n=0}^{\infty}\gamma_n x^n$ at some $n_0\in\mathbb{N}$. In the present case, the recursion relation can be manipulated to the form $\gamma_{n+1}=[c_1(n)/c_2(n)]\gamma_n$, where essentially (since the integral term is in any case bounded from above by λ) c_1 is linear and c_2 quadratic in n. This implies that $\gamma_n\sim(1/n^n)\gamma_0$ and thus the coefficients converge to zero quite rapidly as n increases.

It is worth noting that in principle, the outlined approximation approach is always applicable if the jump component has the geometric form assumed throughout our study and the coefficient functions of the compensated diffusion part are polynomials $\tilde{\mu}(x)=\sum_{i=0}^{N}\tilde{p}_i x^i$ and $(1/2)\sigma^2(x)=\sum_{j=0}^{M}q_i x^i$ such that $q_0=q_1=\tilde{p}_0=0$. Naturally in more general cases the rate of convergence for the coefficient sequence is not necessarily as rapid as in the logistic case.

We apply the outlined procedure to solve the (approximative) optimal thresholds for intensities $\lambda\in\{0.1,1,100\}$ and two different sets of parameters α and β:

(i) symmetric jump size distribution with constant mean 1/2 ($\alpha=\beta$), for $\alpha\in\{1,5,10\}$; as parameter value increases, the distribution becomes more concentrated around its mean; and

(ii) skewed jump size distribution with constant variance 0.01 and variable mean $\overline{z}\in\{0.25,0.5,0.75\}$;

This gives us an illustration of the impact of variable uncertainty in the jump risk with constant "average jump risk" ((i)), and of skewness of the jump size distribution ((ii)). Note that with variance fixed, skewness and mean have opposing effects: for a small mean (which is "good" in the sense that downward jumps are small on average) the distribution is skewed to the right, i.e. towards larger jump sizes, and vice versa. In addition, we will compute the optimal policies in the associated optimization problems for the corresponding continuous (drift-corrected) diffusion $\tilde{X}$. It should be noted that for the associated diffusion instantaneous liquidation is optimal in the problem with discount rate $r + \lambda$ if $\tilde{r} := r+\lambda \geq ab$). The results are given in Tables 6 (symmetric distributions) and 6 (skewed distributions). In both tables, instantaneous liquidation is optimal for $\lambda \in \{1, 10\}$ and hence rows corresponding to $(\tilde{X}, \tilde{r})$ have been omitted in these cases.

λ	α		x^*	y_c^*	η_c^*	x_c^*
0	-	(X, r)	1.003	1.423	0.781	2.378
0.1	-	$(\tilde{X}, r)$	1.263	1.734	0.887	3.011
	-	$(\tilde{X}, \tilde{r})$	0.684	1.134	0.782	1.472
	1	(X, r)	1.053	1.503	0.830	2.469
	6	(X, r)	1.054	1.500	0.824	2.509
	10	(X, r)	1.054	1.500	0.824	2.511
1	-	$(\tilde{X}, r)$	3.546	4.383	1.639	8.594
	1	(X, r)	1.169	1.833	1.166	2.514
	6	(X, r)	1.296	1.931	1.133	2.901
	10	(X, r)	1.307	1.939	1.129	2.947
10	-	$(\tilde{X}, r)$	26.07	29.05	5.932	69.35
	1	(X, r)	1.168	2.699	2.340	2.921
	6	(X, r)	1.356	2.795	2.278	3.164
	10	(X, r)	1.381	2.808	2.270	3.199

Table 1: *Optimal boundaries for the jump diffusion* (35) *and the associated continuous diffusion, when intensity* $\lambda \in \{0.1, 1, 10\}$ *and relative jump size distribution is Beta*(α, α), $\alpha \in \{1, 6, 10\}$.

Inspection of the results shows that the numerical results are in line with our findings: the exercise boundaries for the jump diffusion X are in all cases between the corresponding boundaries for the associated diffusion $\tilde{X}$, provided

λ	$\overline{z}$		x^*	y_c^*	η_c^*	x_c^*
0.1	0.25	$(\tilde{X}, r)$	1.133	1.579	0.835	2.696
		(X, r)	1.019	1.448	0.795	2.430
		$(\tilde{X}, \tilde{r})$	0.551	0.971	0.715	1.204
	0.5	$(\tilde{X}, r)$	1.263	1.734	0.887	3.011
		(X, r)	1.054	1.499	0.823	2.511
		$(\tilde{X}, \tilde{r})$	0.684	1.134	0.782	1.472
	0.75	$(\tilde{X}, r)$	1.392	1.887	0.937	3.324
		(X, r)	1.086	1.555	0.862	2.529
		$(\tilde{X}, \tilde{r})$	0.817	1.294	0.844	1.744
1	0.25	$(\tilde{X}, r)$	2.284	2.932	1.253	5.495
		(X, r)	1.151	1.647	0.906	2.806
	0.5	$(\tilde{X}, r)$	3.546	4.383	1.639	8.594
		(X, r)	1.309	1.941	1.128	2.959
	0.75	$(\tilde{X}, r)$	4.803	5.809	1.982	11.73
		(X, r)	1.192	1.977	1.348	2.557
10	0.25	$(\tilde{X}, r)$	13.57	15.51	3.864	34.68
		(X, r)	1.501	2.401	1.549	3.293
	0.5	$(\tilde{X}, r)$	26.07	29.05	5.932	69.35
		(X, r)	1.387	2.811	2.268	3.208
	0.75	$(\tilde{X}, r)$	38.58	42.42	7.684	105.3
		(X, r)	1.184	3.106	2.824	3.256

Table 2: *Optimal boundaries for the jump diffusion* (35) *and the associated continuous diffusion, when intensity* $\lambda \in \{0.1, 1, 10\}$ *and relative jump size distribution is Beta distributed with variance* 0.01 *and mean* $\overline{z} \in \{0.25, 0.5, 0.75\}$.

that the lower boundaries in question exist (i.e. that *take the money and run* policy is not optimal). From Table 6 one sees that increasingly concentrated jump size distribution seems to lead to higher exercise thresholds for all problems and to a lower dividend size for the impulse control problem. This effect is similar for all sample values of λ, though naturally almost negligible for the smallest sample value and more pronounced for the larger values. It appears from Table 6 that such mononicity does not hold for the case (ii).

7 Concluding comments

In this study we considered the determination of the optimal dividend policy of a risk-neutral firm when the stochastic dynamics of the underlying cash flow are characterizable as a spectrally negative jump diffusion with natural boundaries and geometric jumps. We established a relatively broad set of conditions typically satisfied in most mean-reverting models under which the optimal singular dividend policy is characterizable via the minimal r-excessive map with respect to the underlying jump diffusion. A significant consequence of this representation is that the dynamic dividend optimization problem can be reduced to an equivalent static nonlinear minimization problem. As corollaries of this result we then showed that the associated sequential impulse dividend problem as well as the associated optimal liquidation problem are also solvable in terms of the minimal r-excessive map. In line with previous observations based on continuous cash flow dynamics, the values of these problems were shown to be ordered in an exceptionally strong way: the value of the singular stochastic control problem dominates the value of the associated impulse control problem which, in turn, dominates the value of the associated optimal stopping problem. However, we also demonstrated that the marginal values (and, therefore, Tobin's q associated with these particular problems) are ordered in an analogous way. Hence our results unambiguously indicate that increased policy flexibility has a positive effect on both the value as well as on the marginal value of the optimal policy in the jump diffusion case as well. We also stated a set of typically satisfied conditions under which the values of the considered dividend optimization problems can be sandwiched between the values of two associated dividend optimization problems based on a continuous cash flow dynamics.

Our results generalize the results obtained previously in literature for linear diffusions and demonstrate the strong similarities between the behavior of linear diffusions and spectrally negative jump diffusions with geometric jumps and natural boundaries. From an applied point of view, spectrally negative processes are a very relevant generalization of processes with continuous paths, as they allow the incorporation of discontinuous unanticipated negative shocks into the modeling of the underlying cash flow dynamics. Taking this downside risk into account can be viewed as essential for any model meant to be used in prudent risk management.

While our model allows fairly rich jump structures, as we are reasonably free to choose the distribution of the relative jump sizes, it assumes that the jump component enters the defining stochastic differential equation in geometric form and that the boundaries are natural. It might be of interest to know whether, and to what extent, our results could be extended to encompass more general forms of the jump component and different boundary behaviors. Such extensions are out of the scope of the present study and are, therefore, left for future research.

References

Alvarez, L., 1996. *Demand uncertainty and the value of supply opportunities.* Journal of Economics 64, 163–175.

Alvarez, L., 2001. *Reward functions, salvage values and optimal stopping.* Mathematical Methods of Operations Research 54, 315–337.

Alvarez, L., 2003. *On the properties of r-excessive mappings for a class of diffusions.* Annals of Applied Probability 13, 1517–1533.

Alvarez, L., 2004. *A class of solvable impulse control problems.* Applied Mathematics & Optimization 49, 265–295.

Alvarez, L., Rakkolainen, T., 2006. *A class of solvable optimal stopping problems of spectrally negative jump diffusions.* Aboa Centre for Economics, Discussion Paper No. 9.

Alvarez, L., Virtanen, J., 2006. *A class of solvable stochastic dividend optimization problems: on the general impact of flexibility on valuation.* Economic Theory 28, 373–398 .

Alili, L., Kyprianou, A., 2005. *Some remarks on first passage of Lévy processes, the American put and pasting principles.* Annals of Applied Probability 15:3, 2062–2080.

Avram, F., Palmowski, Z., Pistorius, M., 2006. *On the optimal dividend problem for a spectrally negative Lévy process.* Annals of Applied Probability (to appear).

Bar-Ilan, A., Perry, D., Stadje W., 2004. *A generalized impulse control model of cash management.* Journal of Economic Dynamics & Control 28, 1013–1033.

Bayraktar, E., Egami, M., 2006. *Optimizing venture capital investments in a jump diffusion model.* Preprint.

Bernanke, B. S., 1983. *Irreversibility, Uncertainty, and Cyclical Investment*, Quarterly Journal of Economics, 98:1, 85–103.

Bertoin, J., 1996. *Lévy processes.* Cambridge University Press.

Borodin, A. and Salminen, P. (2002). *Handbook on Brownian motion - Facts and formulae*, 2nd ed. Birkhauser, Basel.

Chan, T., Kyprianou, A., 2006. *Smoothness of scale functions for spectrally negative Lévy processes.* Preprint.

Duffie, D., Pan, J., Singleton, K., 2000. *Transform analysis and asset pricing for affine jump diffusions.* Econometrica 68, 6, 1343–1376.

Gerber, H., Landry, B., 1998. *On the discounted penalty at ruin in a jump-diffusion and the perpetual put option.* Insurance:Mathematics and Economics 22, 263–276.

Gerber, H., Shiu, E., 1998. *Pricing perpetual options for jump processes.* North American Actuarial Journal 2, 3, 101–112.

Gerber, H., Shiu, E., 2004. *Optimal dividends analysis with Brownian motion.* North American Actuarial Journal 8, 1, 1–20.

Jacob, N., Schilling, R., 2001. *Lévy type processes and pseudodifferential operators.* In: O. Barndorff-Nielsen, T. Mikosch and S. Resnick (eds.): *Lévy processes: theory and applications*, Birkhauser, pp. 139–167.

Kyprianou, A., Palmowski, Z., 2006. *Distributional study of De Finetti's dividend problem for a general Lévy insurance risk process.* Preprint.

Mordecki, E., 2002a. *Perpetual options for Lévy processes in the Bachelier model.* Proceedings of the Steklov Mathematical Institute 237, 256–264.

Mordecki, E., 2002b. *Optimal stopping and perpetual options for Lévy processes.* Finance & Stochastics VI 4, 473–493.

Mordecki, E., Salminen, P., 2006. *Optimal stopping of Hunt and Lévy processes.* Preprint.

Perry, D., Stadje W., 2000. *Risk analysis for a stochastic cash management model with two types of customers.* Insurance:Mathematics and Economics 26, 25–36.

Protter, P., 2004. *Stochastic integration and differential equations.* Springer-Verlag, 2nd edition.

Taksar, M., 2000. *Optimal risk and dividend distribution control models for an insurance company.* Mathematical Methods of Operations Research 51, 1–42.

Øksendal, B., 2003. *Stochastic differential equations. An introduction with applications.* Springer-Verlag.

Øksendal, B., Sulem, A., 2005. *Applied stochastic control of jump diffusions.* Springer-Verlag.

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