*Mitri Kitti*

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**ABSTRACT**

Equilibrium payoffs corresponding to subgame perfect equilibria in pure strategies are characterized for infinitely repeated games with discounted payoffs. The equilibrium payoff set of a game is a fixed-point of set-valued operators introduced in the paper. The new operator formalism is utilized in showing the folk theorem for repeated games with unequal but constant discount rates. When the players become more patient, the equilibrium payoff set converges to a particular fixed-point of an asymptotic operator. The limit sets for constant discount rates can be used in analyzing the outer limit of equilibrium payoffs when the discount factors increase but discount rates are not fixed.

JEL Classification: C72, C73

Keywords: repeated game, equilibrium payoff set, folk theorem, unequal discount rates

**Contact information**

Mitri Kitti
Department of Economics
University of Turku
FI-20014, Finland
Email: mitri.kitti (at) utu.fi

## 1. Introduction

The most important theoretical findings for repeated games are the folk theorem for equal discount factors [2, 8] and the fixed-point characterization of equilibrium payoffs [3, 4, 7]. The folk theorem for repeated games tells that any feasible payoff vector that is individually rational is obtained as a subgame perfect equilibrium outcome of a repeated game when the players' have an equal and sufficiently large discount factor. It is well-known that the result changes when the players have unequal discount factors [12]. In general, the limit payoff set is larger when the players have unequal discount factors. The limit set of payoffs for repeated games with unequal discount factors but constant discount rates is characterized in this paper.

The equilibrium payoffs of infinitely repeated games are the largest fixed-point of a set valued operator defined by the players' incentive compatibility constraints, for repeated games with perfect monitoring see [7], for repeated games with imperfect monitoring see [4], and for generalizations to stochastic games see [9, 11]. This characterization result is basically the only one that describes the equilibrium payoffs of repeated games under all circumstances.

In this paper the operator approach is generalized to a family of operators or iterated function systems: there is not only one operator that characterizes the equilibrium payoffs, but infinitely many. The main motivation for these operators comes from the observation that for games with unequal discount factors but constant discount rates, it is possible to define an asymptotic operator, which captures the limiting behavior when the discount factors go to one. The limit payoff set is the smallest fixed-point of the asymptotic operator.

One interpretation of having constant discount rates, while allowing the discount factors to converge to one, is that the time lag between observing the actions and reacting to them vanishes. This would suggest that the limit set equals the set of equilibrium payoffs in a continuous-time repeated game. This is indeed the case for a class of switching strategies defined for continuous-time repeated games when in the limit all action profiles can be played. Switching strategies are studied in [10] and some of the results for the payoffs of these strategies are utilized in this work.

This paper focuses on the pure strategy equilibria under perfect monitoring, i.e., players observe perfectly each others' actions. For the folk theorem type of results it is assumed that if a player would gain from deviating from the minimax action-profile of another player, then the player's own minimax outcome is less than what he would get by minimaxing the other player. The assumption guarantees that for large discount factors the least equilibrium payoffs are the players' minimax payoffs.

It is shown in [6] that in equilibrium it is possible to obtain any payoff corresponding to a path having discounted payoffs arbitrarily little above the minimax level at all time instants, when the discount factors go to one and discount rates remain constant. In essence, the set of payoffs corresponding to the paths that yield payoffs above the minimax levels is given a fixed-point characterization in this paper. Another characterization, given in terms of a fixed-point of a particular operator (different from the asymptotic operator defined in this work) is provided in [15] for the games with imperfect monitoring and correlated strategies.

The case of constant discount rates is useful in analyzing the limit payoffs in the general case of unequal discount factors. An immediate observation on the limit sets for constant discount rates is that in general there is no limit when the discount factors converge to one but the players do not have constant discount rates. This motivates the study of the outer limit set of payoffs when discount factors go to one. This is the limit set obtained by taking the accumulation points of all sequences of equilibrium payoffs in which the players discount factors go to one, i.e., the player become arbitrarily patient but the rates at which their discount factors go to one may differ. It is shown that the outer limit is obtained as the union of the limit sets for constant discount rates, when the ratios of these rates are allowed to become arbitrarily large.

The paper is structured as follows. Section 2 goes through the main concepts used in the paper. The operator formalism for equilibrium payoffs is introduced in Section 3. This formalism is modified to the case of constant discount rates in Section 4. In Section 5 the results for these operators are utilized in showing folk theorem type of results for constant discount rates and the more general case of unequal discount factors. Conclusions are discussed in Section 6.

## 2. Notations and Preliminaries

*2.1. Repeated games and subgame perfection*

There are $n$ players, and $N = \{1, \dots, n\}$ denotes the set of players. The set of actions available for player $i \in N$ in the stage game is $A_i$. Each player is assumed to have finitely many actions. The set of action profiles is denoted as $A = \times_i A_i$. As usual, $a_{-i}$ stands for the action profile of other players than player $i$, and the corresponding set of action profiles is $A_{-i} = \times_{j \neq i} A_j$. Function $u : A \mapsto \mathbb{R}^n$ gives the vector of payoffs that the players receive in the stage game when a given action profile $a$ is played; if $a \in A$ is played, player $i$ receives payoff $u_i(a)$.

The stage game is repeated infinitely many times, and the players discount the future payoffs with discount factors $\delta_i$, $i \in N$. The matrix $T$ will denote the diagonal matrix that has the discount factors $\delta_1, \ldots, \delta_n$ on its diagonal. In Sections 4 and 5 the attention will be on games where $\delta_i = e^{-r_i \Delta}$, $i \in N$, where $r_i > 0$, $i \in N$, are the players discount rates and $\Delta$ is a positive time-step.

Players are assumed to observe the action profile played at the end of each period. A history contains the path of action profiles that have previously been played. The set of length $k$ histories or paths is denoted as $A^k = \times_k A$. The empty path is $\varnothing$, i.e., $A^0 = \{\varnothing\}$. The set of infinitely long paths is denoted by $A^\infty$. The set of paths beginning with a given action profile $a$ are $A^k(a)$.

A strategy for player $i$ in the infinitely repeated game (or the supergame) is a sequence of mappings $\sigma_i^0, \sigma_i^1, \ldots$, where $\sigma_i^k : A^k \mapsto A_i$. The set of strategies for player $i$ is $\Sigma_i$. The strategy profile consisting of $\sigma_1, \ldots, \sigma_n$ is denoted by $\sigma$. Given a strategy profile $\sigma$ and a path $p \in A^k$, $k \geq 0$, the restriction of the strategy profile after $p$ is is $\sigma | p$.

The discounted average payoff for player $i$ corresponding to strategy profile $\sigma$ is

$$U_i(\sigma) = (1 - \delta_i) \sum_{k=0}^{\infty} \delta_i^k u_i \left( a^k(\sigma) \right), \tag{1}$$

where $a^k(\sigma)$ is the action profile in stage $k$ when $\sigma$ is followed. The strategy profile $\sigma$ is a subgame perfect equilibrium (SPE) of the supergame if

$$U_i \left( \sigma | p \right) \geq U_i \left( \sigma_i', \sigma_{-i} | p \right) \text{ for all } i \in N,\ p \in A^k,\ k \geq 0, \text{ and } \sigma_i' \in \Sigma_i.$$

A path that the players can follow in an equilibrium is an induced equilibrium path. An important class of strategies for repeated games are the simple strategies [1].

**Definition 1.** A strategy is simple if it is composed of an initial path $p \in A^\infty$ that is followed until some of the players unilaterally deviates from it. After the deviation is observed the game switches to path $p^i \in A^\infty$, where $i \in N$

corresponds to the deviator. All subsequent deviations lead to paths $p^i$, $i \in N$, corresponding to the deviating player.

The collection of paths followed after deviations, i.e., $\{p^i : i \in N\}$, is called a penal code. The penal code is subgame perfect if the simple strategies derived from it are subgame perfect. These strategies are the ones where the initial path is any of the paths $p^i$, $i \in N$. A penal code is extremal if it is subgame perfect and it leads to the players smallest equilibrium payoffs, i.e., the payoff for player $i$ corresponding to $p^i$ is the smallest SPE payoff in the game. The following result from [1] provides a necessary and sufficient condition for an induced equilibrium path.

**Proposition 1.** *A path $p \in A^\infty$ is an induced equilibrium path if and only if it is supported by an extremal penal code.*

The minimax payoff of player $i$ gives the lower bound of possible payoffs for $i$ in the repeated game. This payoff is denoted as

$$v_i^- = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i \left(a_i, a_{-i}\right).$$

Moreover, $v^-$ denotes the vector composed of players' minimax payoffs. To shorten the notation let us denote

$$d_i(a) = \max_{a_i' \in A_i} u_i(a_i', a_{-i}).$$

The vector $d(a)$ is the vector composed of $d_i(a)$, $i \in N$, i.e., $d(a)$ contains the largest payoffs corresponding to players' unilateral deviations from $a \in A$.

The following assumption is utilized in the analysis of the limit payoffs when the players become more patient.

**Definition 2.** The game has the profitable minimaxing property if there are $a^i \in A$, $i \in N$, such that

1. $u_i(a^i) = v_i^-$ and $u_i(a^i) = d_i(a^i)$,
2. for any $j \neq i$; if $d_j(a^i) > u_j(a^i)$, then $u_j(a^i) > v_j^-$.

The profitable minimaxing property means that the players who would like to deviate from $a^i$ get more than their minimax payoffs. This property guarantees that there is no incentive to deviate from $a^i$ when the punishment yields the minimax payoff and the deviating player's discount factor is large enough.[1]

Let $a^i$ correspond to minimax payoffs for player $i \in N$. The penal code in which $p^i = a^i a^i \cdots$ is called a minimax penal code.

**Proposition 2.** *There are $\bar{\delta}_i$, $i \in N$, such that the minimax penal code is subgame perfect for all $\delta_i \geq \bar{\delta}_i$, $i \in N$, if and only if the game has the profitable minimaxing property.*

*Proof.* Assume that the profitable minimaxing property holds, and let $a^i$ be as in the definition of the profitable minimaxing property. There are no profitable one-shot deviations from the minimax penal code for player $i$ who is minimaxed under the first condition in the definition of the profitable minimaxing property.

Consider players other than $i$. The incentive compatibility condition for player $j$ who is not minimaxed is

$$u_j(a^i) \geq (1 - \delta_j) d_j(a^i) + \delta_j v_j^-. \tag{2}$$

If $d_j(a^i) = u_j(a^i)$, then player $j$ cannot gain by deviating. Consequently, $d_j(a^i)$ is either the minimax payoff of player $j$ or yields payoff above the minimax level, i.e., $d_j(a^i) \geq v_j^-$. This is equivalent to the incentive compatibility condition of player $j$ given that $i$ is minimaxed, and it holds for all $\delta_j \in (0, 1)$.

If $d_j(a^i) > u_j(a^i)$, then $u_j(a^i) > v_j^-$ by the profitable minimaxing property. It follows that $d_j(a^i) - v_j^- > u_j(a^i) - v_j^- > 0$, and the incentive compatibility condition (2) holds if and only if

$$\delta_i \geq \bar{\delta}_i = \frac{d_j(a^i) - u_j(a^i)}{d_j(a^i) - v_j^-}.$$

Note that $\bar{\delta}_i$ is a number smaller than one. Hence, the profitable minimaxing property guarantees that the minimax penal code is subgame perfect for $\delta_i$, $i \in N$, large enough.

If on the other hand, the second condition fails, i.e., $u_j(a^i) \leq v_j^-$, and the minimax penal code still was SPE, the incentive compatibility condition would give $(1 - \delta_j) d_j(a^i) + \delta_j v_j^- \leq v_j^-$, which implies that $d_j(a^j) \leq v_j^-$. To

[1]The profitable minimaxing property is also sufficient for the weak NEU condition [2], and hence implies the folk theorem for equal discount factors.

gather, we would then have $u_j(a^i) < d_j(a^i) \leq v_j^-$. In particular, $u_j(a^i) \neq v_j^-$. This would contradict the subgame perfection of the minimax penal code, because it becomes optimal for player $j$ to deviate and then get the minimax payoff. This proves that if the minimax penal coded is subgame perfect, then it has the profitable minimaxing property. □

Take $T'$ (a diagonal matrix of discount factors) such that all of its diagonal components are no less than those of $T$, i.e., $T' \geq T$, and the discount factors of $T$ are at least $\bar{\delta}_i$, $i \in N$. It holds that if $p \in A^\infty$ is an equilibrium path for $T$, then it is also an equilibrium path for $T'$, see [5]. In other words, monotone comparative statics holds for equilibrium paths.

**Corollary 1.** *Under the profitable minimaxing property the monotone comparative statics hold for induced equilibrium paths when $\delta_i \geq \bar{\delta}_i$ for all $i \in N$.*

The monotone comparative statics of equilibrium paths implies monotone comparative statics for action profiles that can be played in equilibria. Let $A(T)$ denote the set of such action profiles, i.e., $A(T)$ contains all $a \in A$ such that $a$ is on an induced equilibrium path for discount factors $T$. The monotone comparative statics of $A$ means hat $A(T) \subseteq A(T')$ when $T' \geq T$ and the discount factors of $T$ are at least $\bar{\delta}_i$, $i \in N$.

**Corollary 2.** *Under the profitable minimaxing property it holds that $A(T) \subseteq A(T')$ when $T' \geq T$ and the discount factors corresponding to $T'$ and $T$ are no less than $\bar{\delta}_i$ for all $i \in N$.*

Monotone comparative statics for equilibrium actions will be utilized when analyzing the limit of equilibrium payoffs sets for constant discount rates. It follows also from Proposition 2 that when considering the equilibria for constant discount rates, there is $\bar{\Delta}$ such that $\delta_i = e^{-r_i\Delta} \leq \bar{\delta}_i$ for all $\Delta \leq \bar{\Delta}$. The upper bound $\bar{\Delta}$ denotes this upper bound for $\Delta$ throughout this paper.

*2.2. Orbits, trajectories, and feasible payoffs*

To understand the limit properties of the payoff set for constant discount rates some elementary concepts related to the dynamic systems determined by affine mappings are needed. Taking the discounted average utility corresponding to $u(a)$, $a \in A$, and $v \in \mathbb{R}^n$ can be formulated as the operation $DU_a(v) = (I - T)u(a) + Tv$, where $I$ is the $n \times n$ identity matrix. The discounted average of playing $a$ twice and then giving the players the continuation vector $v$ is

$$DU_a^2(v) = DU_a(DU_a(v)) = (I - T^2)u(a) + T^2 v.$$

It follows by induction that

$$DU_a^k(v) = (I - T^k)u(a) + T^k v \quad \text{for all } k \geq 1.$$

The sequence $v^k = DU_a^k(v)$, $k = 1, 2, \ldots$, is the trajectory of $v$ under $DU_a$. Because the limit of this trajectory is $u(a)$, the vector $u(a)$ is included in the trajectory as well.

For the remainder of this section it is assumed that $\delta_i = e^{-r_1\Delta}$, $i \in N$, where $r_1, \ldots, r_n, \Delta > 0$. Let $e^{-\Delta rk}$ denote the matrix that has $e^{-\Delta r_i k}$, $i \in N$, on its diagonal; $T^k = e^{-\Delta rk}$. The trajectories of $v$ lie in the (forward) orbit of $v$. The orbit is the set of all points to which $v$ can be mapped for some choice of $\Delta$. The orbits, as described above, can be viewed as trajectories of the continuous time dynamical system, flow of which is $DU_a^t(v) = (I - T^t)u(a) + T^t v$. We shall return to the case of continuous time later on. The attention is on the limit of equilibrium payoffs as $\Delta$ goes to zero and $r = (r_1, \ldots, r_n)$ are fixed.

The set $V(\Delta)$ denotes the equilibrium payoffs for discount factors corresponding to discount rates $r$ and a given $\Delta$.

**Example 1.** To clarify the above concepts, let us consider the case of two action-profiles and two players. Let $a$ and $b$ stand for the action-profiles. The vector of players' payoffs are $u(a)$ and $u(b)$. For simplicity, assume that $u(a) = (0, 0)$ and $u(b) = (1, 1)$. The players' discount factors are $\delta_1 = e^{-r_1\Delta}$ and $\delta_2 = e^{-r_2\Delta}$. The $2 \times 2$ matrix $T$ has these discount factors on its diagonal.

For the payoff stream in which the payoffs are $u(a)$ for the first $k$ periods and $u(b)$ from stage $k$ onwards, the normalized present value is $T^k u(b)$, this is the trajectory of $u(a)$. It can be seen that these values are on the *orbit* $\phi_{ab}^0$ that passes through the origin and $u(b)$; the orbit is given by $x_2 = x_1^{r_2/r_1}$. When choosing any point $v$ from this set and taking the operation $DU_a v = (I - T)u(a) + Tv$, the outcome is on the same set. Hence, this set is invariant under $DU_a$. In the same way it is possible to describe the orbit $\phi_{ba}^0$ for the operator $DU_b$; $x_2 = 1 - (1 - x_1)^{r_2/r_1}$.

The first observation on orbits is that all the possible payoff vectors that the players can obtain in this simple decision model lie between the two orbits

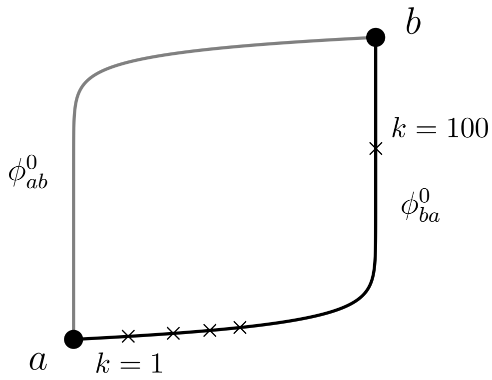


Figure 1: Orbits (solid lines) and points in the trajectory $\phi^{0.01}_{ba}$ (marked with $\times$'s) for $r_1 = 20$, $r_2 = 1$, i.e., player 2 is more patient than player 1.

$\phi^0_{ab}$ and $\phi^0_{ba}$. In particular, when $\Delta$ goes to zero, the region between the two orbits is filled, and in the limit any payoff vector in that region is possible. This idea will be used in showing the folk theorem.

When the question is on an infinitely repeated game, the setup is of course more complicated than in this simple example. However, the main characteristics of the situation remain the same: the possible payoffs are within the region defined by certain orbits and the incentive compatibility constraints.

Next assume that there are $m$ possible stage-game payoffs in the stage-game. It will be seen that the set of all possible payoffs is obtained by taking all orbits that can be created from the stage-game payoff vectors.

**Definition 3.** The orbit of $v$ under $DU_a$ is

$$\phi^0_{av} = \text{cl}\left\{(I - e^{-rt})u(a) + e^{-rt}v : t \geq 0\right\}.$$

Because $t$ can take any value, it does not matter what is $\Delta$. Hence, the orbit is independent of $\Delta$. Moreover, we are only interested in the part of the orbit that is between $a$ and $b$. Hence, it can be assumed that $t \geq 0$. Observe that as $t$ goes to infinity, the point in $\phi^0_{av}$ goes to $a$. By taking the closure this point gets included in the orbit.

**Definition 4.** The trajectory of $v$ under $DU_a$ is

$$\phi^\Delta_{av} = \text{cl}\{w^k : w^k = (I - e^{-\Delta rk})u(a) + e^{-\Delta rk}v,\ k = 1, 2, \ldots\}.$$

Observe that $\phi_{av}^{\Delta} \subseteq \phi_{av}^{0}$. For example, Figure 1 illustrates the first five points in the trajectories $\phi_{ab}^{\Delta}$ and $\phi_{ba}^{\Delta}$. Note that when $v = u(b)$ the orbit $\phi_{av}^{0}$ is denoted as $\phi_{ab}^{0}$, and the corresponding trajectory as $\phi_{ab}^{\Delta}$.

An important observation related to orbits and trajectories is that the distance between two consecutive points in the trajectory is bounded by a term that is proportional to $\max_i(1 - e^{-r_i\Delta})$.

**Remark 1.** The distance between two consecutive points in the trajectory $\phi_{av}^{\Delta}$ is at most $\max_i(1 - e^{-r_i\Delta}) \max_i |u_i(a) - v_i|$.

It follows from the above observation that the Hausdorff distance of an orbit $\phi_{av}^{0}$ to a trajectory $\phi_{av}^{\Delta}$ is at most $\max_i(1 - e^{-r_i\Delta}) \max_i |u_i(a) - v_i|/2$. Note that the Hausdorff distance $d_{\mathcal{H}}$ between $\phi_{av}^{0}$ and $\phi_{av}^{\Delta}$ when $\phi_{av}^{\Delta} \subseteq \pi_{av}^{0}$ is

$$d_{\mathcal{H}}\left(\phi_{av}^{0}, \phi_{av}^{\Delta}\right) = \sup_{x \in \phi_{av}^{0}} \inf_{v' \in \phi_{av}^{\Delta}} \|x - v'\|.$$

The distance between an orbit and a trajectory is the largest when the distance between $v^k$ ($k$'th point in the trajectory) and $v^{k+1}$ takes its largest value. Distance to the trajectory takes its largest value for the point in the middle of $v^k$ and $v^{k+1}$. It follows from this observation that a trajectory can be made arbitrarily close to an orbit by choosing $\Delta > 0$ small enough. This will be an important ingredient in showing the folk theorem.

This far we have only considered orbits of single points $v$. However, the notion of orbit can be further generalized to orbits of sets. In particular, it is possible to first take the orbit of $v$ under $DU_a$, then take all the orbits from points of this set under $DU_b$, and so on. Let $\phi_c^0$, $c = a^m \cdots a^1 \in A^m$, denote the orbits obtained recursively by taking first $\phi_{a^2a^1}^{0}$, then all the orbits from this set under $DU_{a^3}$, i.e., $\phi_{a^3a^2a^1}^{0}$, and so on.

The set of feasible points of the game consists of all the payoffs that are possible in the game. This set is evidently given by taking all the possible orbits recursively as described above

$$FP = \text{cl} \bigcup_{c \in A^m, m \in \mathbb{N}} \phi_c^0.$$

It can be seen that the feasible set is compact.

**Lemma 1.** *$FP$ is a compact set.*

*Proof.* $FP$ is clearly bounded. Hence, it remains to be shown that $FP$ is closed. For this purpose observe that $FP$ is obtained by iterating the operator which takes all the orbits from a given set. To be specific, it is possible to define an operator $\hat{B}$ by setting

$$\hat{B}(S) = \{\phi^0_{ac} : a \in A,\ c \in S\}$$

for a closed set $S$. Iterating $\hat{B}$ for $S^0 = \cup\{u(a) : a \in A\}$ gives a sequence of sets $S^k$ such that $S^k \subseteq S^{k+1}$ for all $k = 0, 1, \ldots$. The sequence has a limit which equals $FP$ and is obtained as a closure of the union of sets $S^k$, $k = 0, 1, \ldots$, (see, e.g., exercise 4.3 in [13]). □

The convex hull of stage-game payoffs is denoted by $FP^0 = \text{conv}\{u(a) : a \in A\}$, and as usual, $IR$ denotes the set of individually rational payoffs:

$$IR = \{v \in \mathbb{R}^n : v \geq v^-\}.$$

Evidently, subgame perfect equilibrium payoffs are feasible and individually rational; they belong to $FP \cap IR$.

In case of equal discount rates, it is well known that when the common discount factor goes to one, the set of equilibrium payoffs converges to $FP \cap IR$, when the least equilibrium payoffs are either equal to the minimax payoffs or they converge to the minimax payoffs. The latter is guaranteed by either full dimensionality or the NEU condition. As will be seen, if discount rates are different the limit set need not be $FP \cap IR$ even when for constant discount rates this would be true.[2] The set of feasible and individually rational payoffs for the case of constant discount rates is denoted as $FP^0 \cap IR$.

In the following example the set $FP$ is illustrated in the prisoners' dilemma game for unequal discount rates

**Example 2.** Consider the payoffs of the prisoners' dilemma game.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $3, 3$ | $0, 4$ |
| $B$ | $4, 0$ | $1, 1$ |

[2]For equal discount factors, profitable minimaxing implies the usual folk theorem, and is therefore sufficient condition for the weak NEU condition which is both necessary and sufficient for the folk theorem when discount factors are equal, see [2].

The set of feasible payoffs is the set enclosed by the dark curves in Figure 2. The payoff vectors in the figure are labeled such that $a = u(B,R)$, $b = u(T,L)$, $c = u(T,R)$, and $d = u(B,L)$. The payoffs in the gray areas are obtained by taking 'higher order' orbits, e.g., taking $\phi^0_{cad}$, and so on.

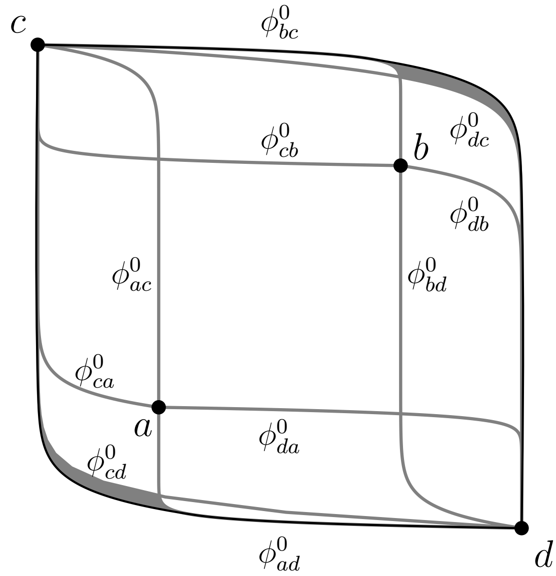


Figure 2: Orbits in the prisoners' dilemma game for $r_1 = 20$, $r_2 = 1$.

## 3. Equilibrium Payoffs

In this section it is shown that SPE payoffs are fixed-point of a class of set-valued monotone operators. The well-known fixed-point theorem of Cronshaw and Luenberger (and Abreu, Pearce, and Stacchetti) [3, 4, 7] is a special case.

### *3.1. Notations*

For any compact set of payoffs $W \subset \mathbb{R}^n$, the smallest payoff for player $i$ in $W$ is

$$v_i^-(W) = \min\{v_i : v \in W\}.$$

Moreover, $v^-(W)$ is the vector composed of $v_i^-(W)$, $i \in N$.

The payoffs $v_i^-(W)$, $i \in N$, will be taken as punishment payoffs that the players get after deviating. It is now possible to test whether the players rather take the actions $a_i$, $i \in N$, at most $k$ times and get $v$ after that than deviate in the first place. The payoff vector corresponding to playing $a$ exactly $j$ times and getting $v$ after that is $(I - T^j)u(a) + T^j v$. Deviating gives at most $(1 - \delta_i)d_i(a) + \delta_i v_i^-(W)$ to player $i \in N$.

**Definition 5.** Let $k$ be a number in $1, 2, \ldots$ and let $W \subset \mathbb{R}^n$ be a compact set. A pair $(a, v)$ of an action profile $a \in A$ and a continuation payoff $v \in W$ is $k$-step admissible with respect to $W$ if for all $j \leq k$ it holds that

$$(I - T^j)u(a) + T^j v \geq (I - T)d(a) + Tv^-(W). \tag{3}$$

Furthermore, $(a, v)$ is $\infty$-step admissible if (3) holds for all $j = 1, 2, \ldots$.

The particular case of $k$-step admissibility when $k = 1$ will be referred to as admissibility. Note that $k$-step admissibility implies $(k-1)$-step admissibility, and so on. The incentive compatibility constraint (3) says that player $i$ rather takes action $a_i$ at most $k$ times and then gets the payoffs $v_i$, than deviates at any of the $k$ stages and then obtains $v_i^-(W)$. Note that the left hand side of (3) is $DU_a^j(v)$ in the notation introduced in Section 2.2.

For a given compact set $W$ let us define a point to set mapping $B_a^k$, $a \in A$, from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ as

$$B_a^k(v; W) = \left\{ DU_a^j(v) : (a, v) \text{ is } j\text{-step admissible w.r.t. } W,\ j \leq k \right\},$$

and for $k = \infty$

$$B_a^\infty(v; W) = \text{cl}\left\{ DU_a^j(v) : (a, v) \text{ is } j\text{-step admissible w.r.t. } W,\ j \in \mathbb{N} \right\}.$$

By taking the closure, the limit point of $(I - T^k)u(a) + T^k v$, $k = 1, 2, \ldots$, i.e., $u(a)$ is included into $B_a^\infty(v; W)$ if $(a, u(a))$ is admissible. Note that $B_a^k(v; W)$ contains the first $j^*$ points on the trajectory of $v$ under $DU_a$, where $j^*$ is the largest number $j \leq k$ for which $(a, v)$ is $j$-step admissible. The mapping $B_a^\infty(v; W)$ contains the maximal number of points on the trajectory of $v$ under $DU_a$ such that each point is of the form $DU_a^j(v)$ for a $j$-step admissible pair $(a, v)$ and $j \in \mathbb{N}$.

For a set of payoffs $W$, the operator $B_a^k(W)$ is the union of $B_a^k(v; W)$ over $v \in W$. The $k$-step operator for a payoff set $W$ is

$$B^k(W) = \bigcup_{a \in A} B_a^k(W).$$

It is worth observing that $B_a^k(W)$ can be an empty set, which only means that $a$ cannot be played if the continuations are from the set $W$. Hence, the union over $A$ in the definition of $B^k$ could be taken instead $A$ over those action profiles for which there are $v \in W$ such that $(a, v)$ are 1-step admissible.

Note that $k$ in the definition of $B^k$ is allowed to be infinitely large, in which case we get the operator $B^\infty$. As will be seen, this operator is particularly important in the analysis of equilibrium payoffs, when the time-step $\Delta$ goes to zero while the discount rates are fixed.

*3.2. Results for k-step operators*

The first observation on $B^k$ is that it is monotone and preserves compactness.

**Lemma 2.** *For any $k \geq 1$, $B^k$ is monotone, $B^k(W^1) \subseteq B^k(W^2)$ if $W^1 \subseteq W^2$, and preserves compactness.*

*Proof.* It can be seen that each $B_a^k$, $a \in A$, preserves compactness. Hence, $B^k$ preserve compactness, too. Moreover, each $B_a^k$, $a \in A$, is monotone, which implies that $B^k$ is monotone, too. □

An important result that will be utilized in this paper is that any set that generates itself under $B^k$ is a subset of equilibrium payoffs. This result is a generalization of the self-generation result for $B^1$, see [3, 4, 7].

**Proposition 3.** *If $W \subseteq B^k(W)$, then $W$ belongs to the set of subgame perfect equilibrium payoffs $V$.*

*Proof.* The result follows by showing that $W \subseteq B^k(W)$ implies that for any $v \in W$ it is possible to create a path such that there are no one shot deviations from it at any stage given that deviations are punished with paths that yield $v_i^-(W)$ for the deviating player $i \in N$. In particular, this construction can be made for the payoff vector $v$ corresponding to $v_i^-(W)$. Due to the one-shot deviation principle the resulting path is an equilibrium.

Let us go through the construction of the path. Because $v \in B^k(W)$ ($k = 1, 2 \ldots$ or $k = \infty$), there are $a^1 \in A$ and $v^1 \in W$ such that $a$ can be played at most $k$ times, let us say $k_1 \leq k$ times such that no player is willing to deviate given that they get the continuation $v^1$ after $k_1$ periods. Let $(a^i)^k$ denote a path in which $a^i$ is played $k$ times. The same argument can be repeated for $v^1 \in W$ and so on ad infinitum, which gives a path $(a^1)^{k_1}(a^2)^{k_2}(a^3)^{k_3}\cdots$. No player has an incentive to deviate from this path at any stage given that the deviations are punished with a path that yields $v_i^-(W)$ for the deviating player $i \in N$. In particular this construction can be done for $v \in W$ with $v_i = v_i^-(W)$ for some $i \in N$. Hence, the payoffs in $W$ correspond to subgame perfect equilibria. □

The next result tells that the set of equilibrium payoffs corresponding to given discount factors corresponding to $T$ is the largest fixed-point of any of the operators $B^k$. Here, the largest fixed-point refers to the largest set in terms of set inclusion. In mathematical terms, it could be said that SPE payoffs are the largest fixed-point of the iterated function system corresponding to $B_a^k$, $a \in A$, and the incentive compatibility constraints.

**Proposition 4.** *For any $k \geq 1$ or $k = \infty$, the set of subgame perfect equilibrium payoffs $V$ is the largest fixed-point of $B^k$ in the class of compact sets.*

*Proof.* The result follows directly from Proposition 3 and the monotonicity of $B^k$. Namely, any $v \in V$ is supported by an extremal penal code, which implies that $v \in B^k(V)$. Hence, $V$ is a fixed-point of $B^k$. By the monotonicity it has to be largest fixed-point in the set inclusion. □

The following result tells that the fixed-point iteration converges $V$ when the initial set is sufficiently large. The convergence of $W^k$ to $V$ refers to Painlevé-Kuratowski convergence, which in the case of compact sets is equivalent to the convergence in the Hausdorff metric, see, e.g., [13].

**Proposition 5.** *The fixed-point iteration*

$$W^{j+1} = B^k(W^j),\ j = 0, 1, \ldots, \tag{4}$$

*converges to $V$ when $W^0$ is a compact set that contains $V$.*

*Proof.* First, observe that $V \subseteq B^k(W^T)$ for all $T = 0, 1, \ldots$ because $B^k$ is monotone and $V \subseteq W^0$. Moreover, it holds that $B^k(W^0) \subseteq W^0$, i.e., $W^1 \subseteq W^0$. The monotonicity of $B^k$ implies that $W^2 = B^k(W^1) \subseteq W^1$. By induction $W^{j+1} \subseteq W^j$ for all $j = 0, 1, \ldots$. Note that the sets $W^j$, $j = 0, 1, \ldots$, are compact (Lemma 2). The limit of the iteration is $V^* = \cap_j W^j$. The set $V^*$ is a fixed-point of $B^k$, because $V^* \subseteq B^k(V^j)$ for all $j = 0, 1, \ldots$. In particular $V^* \subseteq B^k(V^*)$ which implies that $V^*$ is a subset of $V$. On the other hand, $V$ is contained in $B^k(V^k)$ for all $j$, which implies that $V^*$ contains $V$. Hence, $V = V^*$. □

Because $FP$ is a compact set by Lemma 1, the iteration (4) converges when the initial sets is the set of all feasible payoffs.

**Corollary 3.** *The iteration (4) converges to $V$ when the initial set is $FP$.*

## 4. Games with Constant Discount Rates

In this section the focus is on the pure strategy equilibria under constant discount rates and on the relevant operators for the analysis of equilibrium payoffs of such games.

*4.1. Operator formalism for constant discount rates*

The operator formalism as such does not require any assumption on the stage game. However, because the focus is on the limit of equilibrium payoffs when $\Delta$ goes to zero, let us assume the profitable minimaxing property at this point. Corresponding to the discount factors $\bar{\delta}_i$, $i \in N$, above which there is a minimax penal code, it is possible to find $\bar{\Delta}$ such that the discount factors $\delta_i = e^{-r_i\Delta}$ are at least $\bar{\delta}_i$ for all $i \in N$. The operators will be defined assuming that $\Delta \leq \bar{\Delta}$, which means that the players smallest equilibrium payoffs are the minimax payoffs $v_i^-$, $i \in N$. Recall that $V(\Delta)$ denotes the set of equilibrium payoffs as a function of $\Delta$ when the discount rates are fixed.

Let $B_a^\Delta$ denote the operator $B_a^\infty$ for $\delta_i = e^{-\Delta r_i}$. Because $B_a^\Delta(v) \in \phi_{av}^\Delta$ and $v^-(V(\Delta)) = v^-$ for $\Delta \leq \bar{\Delta}$, there is another expression for $B_a^\Delta$.

**Remark 2.** When $\Delta \leq \bar{\Delta}$, it holds that

$$B_a^\Delta(W) = \{w \in \phi_{av}^\Delta : \; v \in W, \text{ and } w \geq (I - T)d(a) + Tv^-\}. \tag{5}$$

Note that there is a minor difference in the definition of $B_a^k$ and $B_a^\Delta$; the payoff $v^-(W)$ in the admissibility condition (3) is replaced with the vector of minimax payoffs. Moreover, the incentive compatibility condition in expression (5) means that $w_i = (I - e^{-r_i\Delta k})u_i(a) + e^{-r_i\Delta k}v_i \geq (I - e^{-r_i\Delta})d_i(a) + e^{-r_i\Delta}v_i^-$ for all $i \in N$, i.e., $(a, v)$ is $k$-step admissible. Recall that $k$-step admissibility implies that $(a, v)$ is also $j$-step admissible for all $j \leq k$.

For any $\Delta > 0$ the operator $B^\Delta$ is defined as the union of $B_a^\Delta$;

$$B^\Delta(V) = \bigcup_{a \in A} B_a^\Delta(V).$$

It can be seen that $B^\Delta$ has the same properties as $B^k$. In particular, it is monotone.

By the definition of $B^\Delta$, the set $V(\Delta)$ contains points that are feasible. Moreover, the payoffs are also individually rational due to the incentive compatibility constraint in the definition of the operator. As will be seen later, for unequal discount rates it may happen that some feasible and individually rational payoffs are not reached even when $\Delta$ goes to zero.

*4.2. The asymptotic operator*

Before considering asymptotic operator, the operator obtained when $\Delta$ goes to zero, let us first make some observations on the set of action profiles that can be played in equilibria. The set $A(\Delta)$ denotes the set of possible action profiles $A(T)$ when $\delta_i = e^{-r_i\Delta}$, $i \in N$. To be specific;

$$A(\Delta) = \{a \in A : (a, v) \text{ is admissible w.r.t. } V(\Delta) \text{ for some } v \in V(\Delta)\}.$$

It can immediately be observed that $A(\Delta)$ is monotone for $\Delta$ small enough. Note that $\bar{\Delta}$ is the upper bound for $\Delta$ corresponding to the discount factors above which the minimax penal code is an equilibrium

**Remark 3.** $A(\Delta^1) \subseteq A(\Delta^2)$ when $\Delta^2 \leq \Delta^1 \leq \bar{\Delta}$.

In important implication of the monotonicity is that $A(\Delta)$ converges when $\Delta$ goes to zero. This limit set is denoted by $A^*$. Note that $A$ in the definition of $B^\Delta$ can be replaced with $A^*$.

In the limit when $\Delta$ goes to zero, the trajectory $\phi_{av}^\Delta$ goes to the orbit $\phi_{av}^0$ in the Hausdorff metric. Hence, for the case of $\Delta = 0$, the asymptotic operator is defined by replacing $\phi^\Delta$ in the definition of $B^\Delta$ with $\phi^0$. Moreover, in the continuous time limit the incentive compatibility condition becomes simply $v \geq v^-$. It can also be observed that if $(I - e^{-rt})u(a) + e^{-rt}v \geq v^-$ for $v \geq v^-$, then $(I - e^{-r\tau})u(a) + e^{-r\tau}v \geq v^-$ for all $0 \leq \tau \leq t$. Recall that the profitable minimaxing property guarantees that the minimax payoffs are attained for $\Delta$ small enough. Hence, the "limit" of $B_a^\Delta$, for $a \in A^*$, when $\Delta$ goes to zero can be defined as

$$B_a^0(W) = \{w \in \phi_{av}^0;\ w \geq v^-, v \in W\},$$

and by taking the union over $A^*$ we get the asymptotic operator

$$B^0(W) = \bigcup_{a \in A^*} B_a^0(W).$$

Because the payoff vectors corresponding to the minimax payoffs are equilibria for small enough $\Delta$, it is convenient to set

$$B^0(\emptyset) = \{u(a^i) : a^i \text{ is as in Definition 2, } i \in N\}.$$

The intuition behind the asymptotic operator is that it defines payoff obtained when $a \in A^*$ are played as long as possible when given the continuation payoffs $w \in W$. The resulting payoffs belong to $\phi^0_{aw}$ for $a \in A^*$ and $w \in W$.

Unlike $B^\Delta$, the operator $B^0$ has the property that $W \subseteq B^0(W)$ for any $W$ such that $v \geq v^-$ for all $v \in W$. This is simply because setting $t = 0$ gives $DU_a^t(v) = v$ for any $v \in IR$. Consequently, $B^0$ does not have a largest fixed-point. However, there is a particular fixed-point that is of interest. To define this fixed-point set some additional definitions are needed.

First, for an action profile to be relevant in generating any new payoffs from a compact set $W$ there should be $v \in W$ and $t > 0$ such that $DU_a^\tau(v) \geq v^-$ for all $\tau \in [0, t]$. The maximal sets of relevant action profiles in terms of set inclusion are of interest. Related to such a set $A' \subseteq A$ we can define the space of closed sets from which the continuations can be taken. Formally this space is as defined below, see also [10].

**Definition 6.** A compact set $S \subseteq IR$ belongs to $\mathcal{C}(A')$, $A' \subseteq A$, if for any $a \in A'$ there is $v \in S$ and $t > 0$ such that $DU_a^\tau(v) \geq v^-$ for all $\tau \in [0, t]$.

The fixed-point set of $B^0$ that will be shown to equal the limit of repeated game payoffs is $V(0)$ and is defined below.

**Definition 7.** $V(0)$ is the smallest fixed-point of $B^0$ such that $V(0) \in \mathcal{C}(A^*)$.

The existence of $V(0)$ follows from the monotonicity of $B^0$ and the profitable minimaxing property, see [10], where $V(0)$ is related to the equilibria of continuous-time repeated games.

## 5. Limit Results for Unequal Discount Factors

### *5.1. Folk theorem for constant discount rates*

Because the operator $B^0$ can be seen as the limit of $B^\Delta$ when $\Delta$ goes to zero, we can expect the set $V(\Delta)$ to converge to $V(0)$. This is shown in the following result, which can be interpreted as a folk theorem for games that satisfy the profitable minimaxing property. The proof is presented in the Appendix.

**Proposition 6.** *Under the profitable minimaxing property $V(\Delta)$ converges to $V(0)$, when $\Delta$ goes to zero.*

In general the set $V(\Delta)$ differs from $V(0)$ for all $\Delta > 0$. To see the reason for this, consider Example 1. No matter how small $\Delta > 0$ is chosen, the trajectory $\phi^{\Delta}_{ab}$ is never exactly equal to the orbit $\phi^{0}_{ab}$. In particular, for any two points from $\phi^{\Delta}_{ab}$, there will be a gap between them along $\phi^{0}_{ab}$. These gaps do not disappear when iterating the operator $B^{\Delta}$.

Another observation related to $V(0)$ is that it is not necessarily equal to the set of feasible and individually rational payoffs. In Figure 5.1 this is demonstrated for the prisoners' dilemma game.

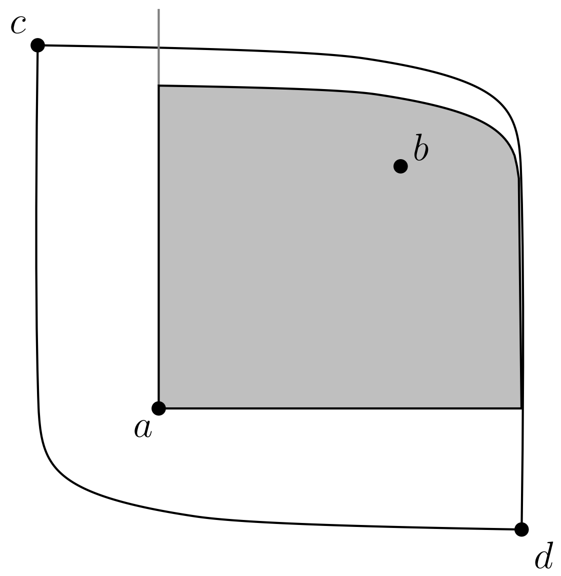


Figure 3: Limit set $V(0)$ in the prisoners' dilemma game for $r_1 = 20$, $r_2 = 1$.

In addition to characterizing the limit payoffs it is possible to characterize the action profiles that can be played in equilibrium, i.e., the set $A^*$. What is important in the following result is that $A^*$ is determined without referring to $V(0)$. For the proof see Lemma 8 in the Appendix.

**Proposition 7.** *Let $N'$ be the set of players $i \in N$ for whom $v_i^-$ is the only payoff in $FP^0 \cap IR$. Under the profitable minimaxing property $A^* = A$ if $N' = \emptyset$ and otherwise*

$$A^* = \{a \in A : u_i(a) = d_i(a) = v_i^- \text{ for all } i \in N'\}.$$

An important imlication of the above result is that the limit of repeated game payoffs is the same as the equilibrium payoffs in continuous-time repeated game when $A^* = A$, i.e., $N' = \emptyset$. If $N' \neq \emptyset$ there can be more payoffs in the continuous-time game. The following example demonstrates this.

**Example 3.** In this game there are three players. However, the third player has only one action.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $0, 0, 1$ | $-1, 1, 1$ |
| $B$ | $1, -1, 1$ | $0, 0, 0$ |

The minimax payoffs are attained at $a = (B, R)$ and $v^- = (0, 0, 0)$. The only action profile that can be taken is $a$. Note in particular that $b = (T, L)$ cannot be played because both player 1 (the row player) and player 1 (the column player) prefer deviating from it. However, in the continuous-time game with immediate reactions to deviations neither of the two players would gain by deviating from $b$. Hence, in a continuous-time repeated game $b$ can be played. Because $u_3(b) = 1$, the payoffs in the continuous-time game would be anything between 0 and 1 for the third player.

The limit results are presented for games where the players' smallest equilibrium payoffs are the minimax payoffs due to profitable minimaxing property. However, if the least equilibrium payoffs converge when $\Delta$ goes to zero, then all the results directly extend to such games with $v_i^-$, $i \in N$, replaced with the limits of players' smallest payoffs. In general, these limit payoffs may differ from the minimax payoffs, see, e.g., [16].

*5.2. The outer folk theorem*

In this section the focus is on the outer limit of equilibrium payoffs when the discount factors converge to one but are allowed to be different. Recall that $T$ denotes the diagonal matrix of discount factors. Moreover, let $V(T)$ denote the set of equilibrium payoffs for constant discount factors corresponding to $T$. The outer limit refers to the set that is obtained by letting $T$ converge to the identity matrix $I$ from below and taking all limit points of sequences in which payoffs belong to $V(T)$ for different $T$, see, e.g., [13] for the definition of the outer limit (or the upper limit). This is the set

$$V^* = \limsup\{V(T) : T \to_- I\}.$$

The interior of the outer limit set gives all the possible limit point that can be obtained when the players become more patient, but the rates at which they become more patient are allowed to differ.

In the following $V(0;r)$ denotes the limit of equilibrium payoffs for given vector $r > \mathbf{0}$, i.e., $r_i > 0$ for all $i \in N$, discount rates. The results for constant discount rates are crucial for understanding the limiting behavior when the discount rates converge to one. The first observation is that there need not be a limit set for $V(T)$ when $T$ converges to $I$ from below, because $V(0;r)$ may differ for different vectors of discount rates; corresponding to different discount rates the limit set is different, which means that there is no limit in the sense of usual set convergence. However, the set $V^*$ is always well-defined.

The outer limit can be obtained by considering the payoffs for constant discount rates. To be specific, all points in $V^*$ are obtained by considering the case when the ratios of the discount rates converge.

**Lemma 3.** *Any $v \in V^*$ corresponds to a sequence of payoffs $\{v^k\}_k$ for which either $\log(\delta_i^k)/\log(\delta_j^k)$ or $\log(\delta_j^k)/\log(\delta_i^k)$ has limit for all $i \neq j$, $i, j \in N$.*

*Proof.* Take a sequence $\{v^k\}_k$ with $v^k \in V(T^k)$, $k = 1, 2, \ldots$, such that the sequence converges to $v$. Pick a subsequence such that either $\log(\delta_i^k)/\log(\delta_j^k)$ or $\log(\delta_j)/\log(\delta_i)$ has a limit for each pair of players $i$ and $j$, $i \neq j$. Namely, if $\log(\delta_i^k)/\log(\delta_j^k)$ is not bounded then $\log(\delta_j)/\log(\delta_i)$ has one limit point equal to zero. The payoff corresponding to the limit of this subsequence is $v$. □

To understand what happens for the payoffs for different discount rates, let us consider the comparative statics of $V(0;r)$ for $r$. The first result relies on the fact that an orbit $\phi_{av}$ is the same whenever the ratios of the discount rates remain the same. Hence, $B_a(\cdot;r)$, only depends on the ratios of the discount rates. For the proof see [10].

**Lemma 4.** *$V(0;r)$ is homogenous of degree zero as a function of $r$; $V(0;\lambda r) = V(0;r)$ for all $\lambda > 0$ and $r > \mathbf{0}$.*

As is stated in the following lemma (see [10]), the set of equilibrium payoffs increases when the ratios of the discount rates increase. In the following $\pi = i_\pi^1 i_\pi^2 \cdots i_\pi^n$ denotes a permutation of players and $R(\pi)$ vectors of discount rates such that $r_{i_\pi^j} \geq r_{i_\pi^{j+1}} > 0$ for all $j = 1, \ldots, n$. Moreover, $\Pi(N)$ stands for the set of all permutations of players.

**Lemma 5.** *If $r$ and $q$ are vectors of discount rates belonging to $R(\pi)$ such that $r_{i^j_\pi}/r_{i^{j+1}_\pi} > q_{i^j_\pi}/q_{i^{j+1}_\pi}$ for all $j = 1, \ldots, n-1$, then $V(0; q) \subset V(0; r)$.*

As a corollary of the above result, it follows that the set of feasible and individually rational payoffs belong to $V(0; r)$ for all discount rates $r$. Recall that $FP^0 \cap IR$ contains all the points obtained by randomizing over $\{u(a) : a \in A\}$ that satisfy $v \geq v^-$. By the usual folk theorem, $FP^0 \cap IR$ is obtained in the limit when the discount factors are equal and converge to one.

**Corollary 4.** *$FP^0 \cap IR \subseteq V(0; r)$ for all vectors of discount rates $r$.*

The following result tells that the outer limit $V^*$ can be obtained by taking the union of all the limit sets for fixed discount rates.

**Proposition 8.** *Under the profitable minimaxing property it holds that*

$$V^* = \bigcup_{\pi \in \Pi(N)} \lim \left\{ V(0; r) : r \in R(\pi),\ \frac{r_{i^j_\pi}}{r_{i^{j+1}_\pi}} \to \infty \textit{ for all } j = 1, \ldots, n-1 \right\}.$$

*Proof.* Lemma 3 implies that $V^*$ is obtained by taking the union all the limits of payoffs when the ratios $\log(\delta_i^k)/\log(\delta_j^k)$, $i \neq j$ converge. For fixed ratios, i.e., fixed discount rates $r_1, \ldots, r_n$ the limits are obtained by considering the limits of $V(0; r)$, i.e., the equilibrium payoffs in the continuous time case for discount rates $r$. It follows from lemmas 4 and 5 that it is enough to consider the limits of these set for all possible permutations of players, when the ratios of the discount rates become arbitrarily large. Moreover, the monotone comparative statics of $V(0; r)$ for $r$, i.e., Lemma 5, implies that $\lim V(0; r)$ exists when $r_{i^j_\pi}/r_{i^{j+1}_\pi} \to \infty$ for all $j = 1, \ldots, n-1$. □

**Example 4.** Consider the prisoners' dilemma game of Example 2. The outer limit in this game is the union of boxes $[1, 4] \times [1, 11/3]$ and $[1, 11/3] \times [1, 4]$ that is illustrated in Figure 4. Note that the orbit $\phi_{dv}$ converges to the set that is composed of lines from $u(d)$ to $(u_1(d), v_2)$ and from $(u_1(d), v_2)$ to $v$, when $r_1/r_2$ goes to infinity. As a consequence, the limit of $V(0; r)$ is $[1, 4] \times [1, 11/3]$ when $r_1/r_2$ goes to infinity. It can be seen in this example that the usual conclusion of the folk theorem for equal discount factors does not hold even for the outer limit of the equilibrium payoffs. In this example the outer limit of all feasible payoffs is $[0, 4] \times [4, 0]$ (region inside the dotted lines in the figure), the intersection of this set with the individually rational payoffs is $[1, 4] \times [1, 4]$, which is not equal to the union of $[1, 4] \times [1, 11/3]$ and $[1, 11/3] \times [1, 4]$.

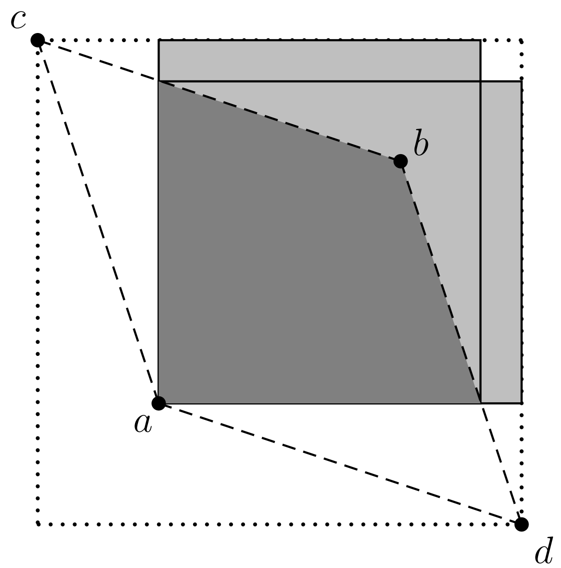


Figure 4: The feasible and individually rational points for equal discount factors (gray area), the limit set $V^*$ (light gray area), and the outer limit of all feasible points (region inside the dotted square) in the prisoners' dilemma game.

## 6. Conclusions

The main contribution of this paper is a new operator formalism for analyzing repeated games with discounted payoffs. The formalism enables both the computation of equilibria and the analysis of the equilibrium payoffs when the players discount factors converge to one. In particular, it possible to obtain a folk theorem type of limit result for games with unequal discount factors.

A central concept for games with constant discount rates is the asymptotic operator: the equilibrium payoffs converge to a fixed-point of the asymptotic operator, when the players become more patient and have constant discount rates. This set is typically smaller than the set of all feasible and individually rational payoffs in the game.

The limit set of a game with fixed discount rates is the set of equilibrium payoffs in a continuous-time game with no time-lags [10]. This gives another interpretation for the folk theorem: the limit set is obtained when the time-lag between observing deviations and reacting to them becomes arbitrarily small.

The case of constant discount rates is useful in analyzing the general case of unequal discount factors. For such games the payoffs may not converge, while there always exists an outer limit for the equilibrium payoffs. This limit can be interpreted as the set of all possible payoffs that can be obtained, when the discount factors converge to one in an arbitrary manner. The outer limit differs considerably from the set of feasible and individually rational payoffs in the case of equal discount factors. The outer limit is the union of all limit sets corresponding to different cases of fixed discount rates. This set contains the set of feasible and individually rational payoffs of a repeated game with constant discount factors.

## Appendix A. Proof of the main result

The following results are utilized in the proof of Proposition 6. The first shows that the set $V(0)$ is found by fixed-point iteration. The second shows that the limit set $A$ is the same regardless of discount rates. The last one characterizes $A^*$ and shows that there is a path $p$ with payoff $v(p,\Delta)$ for time step $\Delta$ such that $(a, v(p,\Delta))$ is admissible for any $a \in A^*$, and $v(p,\Delta)$ converges when $\Delta$ goes to zero.

**Lemma 6.** *The fixed-point iteration*

$$W^{k+1} = B^0(W^k). \tag{A.1}$$

*converges to* $V(0)$ *when* $W^0 \in \mathcal{C}(A^*)$ *and* $W^0 \subseteq V(0)$.

*Proof.* The result follows by observing that for any $W^0 \in \mathcal{C}(A^*)$ such that $W^0 \subseteq V(0)$, the iteration is bounded above by the smallest fixed-point of $B$ in $\mathcal{C}(A^*)$. Due to monotonicity of $B^0$ it holds that $W^k \subseteq W^{k+1}$, which means that the limit of the iteration in the space $\mathcal{C}(A^*)$ is the closure of the union of $W^k$, $k \geq 0$. This set is a fixed-point of $B^0$ and hence equals $V(0)$. □

**Lemma 7.** *The limit set* $A^*$ *is the same for all discount rates.*

*Proof.* Let $r^*$ stand for the smallest discount rate $\min_i r_i$. By the comparative statics of equilibrium paths (Corollary 1), if $a$ can be played in a game where the discount factors are $e^{-r^*\Delta}$, it can also be played in the game where the discount factors are $e^{-r_i\Delta}$, $i \in N$, and the time step is $\Delta$. On the other hand, if $a \in A$ can be played for discount rates $r$ and time step $\Delta$, then it can also be played when the discount rate is $r^*$ for all players and time step is $\Delta^* = \max_{i\in N} r_i\Delta/r$. Hence, the limit of $A(\Delta)$ is the same as the one obtained for constant discount rates. In particular, it is the same regardless of discount rates. □

The following result shows that there is a path of action profiles $p$ that gives a payoff vector $v(p, \Delta)$ such that $(a, v(p, \Delta))$ is admissible for each $a \in A^*$. The latter property is utilized when choosing the initial set $W^0$ for the iteration (A.1) in the proof of Proposition 6. Moreover, the set $A^*$ is characterized. In the following $N'$ is the set of players $i \in N$ for whom $v_i^-$ is the only payoff in $FP^0 \cap IR$.

**Lemma 8.** *Under the profitable minimaxing property $A^* = A$ if $N' = \emptyset$ and otherwise*

$$A^* = \{a \in A : u_i(a) = d_i(a) = v_i^- \text{ for all } i \in N'\}.$$

*For $\Delta$ small enough there is an SPE path $p$ such that $(a, v(p, \Delta))$ is admissible for any $a \in A^*$, and $v(p, \Delta)$ converges when $\Delta$ goes to zero.*

*Proof.* Let $\hat{A}$ be the $A$ if $N' = \emptyset$ and otherwise

$$\{a \in A : u_i(a) = d_i(a) = v_i^- \text{ for all } i \in N'\}.$$

Let us also set $S = \text{conv}\{u(a) : a \in \hat{A}\} \cap IR$. If this set is a singleton, all the action profiles in $S$ the same payoff vector. Otherwise it has a non-empty relative interior. In the first case, the unique payoff $v$ in $S$ is such that $v_i = u_i(a) = d_i(a) = v_i^-$ for all $i \in N$ and for some $a \in A$. Assume the latter case and that $v$ is in the relative interior of $S$ and corresponds to a convex combination $\sum \lambda_j u(a^j)$, $\sum_j \lambda_j = 1$ and $\lambda_j \geq 0$, $j = 1, \ldots, k$.

It is possible to get arbitrarily close to values $\lambda_j$, $j = 0, \ldots, k$, by choosing $K$ large enough and weights $\hat{\lambda}_j = n^j/K$, $\sum n^j = K$. In particular, for $K$ large enough the resulting vector $\sum_j \hat{\lambda}_j u(a^j)$ is in the relative interior of $S$. Consider the sequence $p(K)$ in which $a^1$ is played $n^1$ times, after which $a^2$ is played $n^2$ times and so on until $a^k$ is played $n^k$ times. Assume that this sequence is repeated, let $p^\infty(K)$ denote the resulting path of action profiles. The path $p^\infty(K)$ corresponds to an infinite sequence of payoffs that has a Césaro limit; the limit of means of the resulting payoff sequence is

$$v^K = \frac{1}{K} \sum_j n^j u(a^j).$$

It follows from a Tauberian theorem (see, e.g., Appendix A.4 in [14]) that the limits of payoffs corresponding to $p^\infty(K)$ exist for all players when $\Delta$ goes to zero, and these limits are the elements of the vector $v^K$. Moreover, this vector is the limit no matter at which stage of the path the game is started.

Let $v^{K,j}(\Delta)$, $j = 0, \ldots, K$, denote the discounted payoff vectors corresponding to the path $p^\infty(K)$ when starting from stage $j = 0, \ldots, K$. Because $p^\infty(K)$ corresponds to infinite repetition of $p(K)$, these are all the discounted payoffs there are along the sequence $p^\infty(K)$. By choosing $\Delta$ close enough to zero all these vectors can be made arbitrarily close to $v^K$. In particular, we can first choose $K$ large enough such that $v^K$ is on the relative interior of $S$ and then $\Delta$ small enough such that $v^{K,j}(\Delta)$, $j = 0, \ldots, K$, are also on the relative interior of $S$, and

$$(I - e^{-r\Delta})u(a^j) + e^{-r\Delta}v^{K,j+1} \geq (I - e^{r\Delta})d(a) + e^{-r\Delta}v^-,$$

where $v^{K,K+1} = v^{K,0}$. Hence, for $\Delta$ small enough $p^\infty(K)$ can be played in equilibrium.

Because $v^{K,j}$, $j = 1, \ldots, K$, are in the relative interior of $S$, it follows that any pair $(a, v^{K,j})$, $a \in \hat{A}$, $j = 0, \ldots, K$ becomes admissible when $\Delta$ is close enough to one, i.e., for any $a \in \hat{A}$ and $j = 0, \ldots, K$ it holds that

$$(I - e^{-r\Delta})u(a) + e^{-r\Delta}v^{N,j} \geq (I - e^{r\Delta})d(a) + e^{-r\Delta}v^-,$$

for $\Delta$ small enough. Hence, $p^\infty(K)$ is suitable path $p$ for which $(a, v(p, \Delta))$, $a \in A$ are admissible and $v(p, \Delta)$ converges to a vector $v$ when $\Delta$ goes to zero. This proves the last claim of the lemma.

Note that for equal discount rates $S = FP^0 \cap IR$. Hence, all action profiles in $\hat{A}$ can be played in the limit when players have equal discount rates. If the players have equal discount rates, and for some player $i$ the only payoff in $FP^0 \cap IR$ is $v_i^-$, then the only action profiles that this player can accept are the ones that satisfy $u_i(a) = d_i(a) = v_i^-$. It follows that an action profile $a \notin \hat{A}$ cannot be played, which assures that the limit of $A(\Delta)$ for equal discount rates is $\hat{A}$. Lemma 7 implies that $\hat{A} = A^*$. □

*Proof of Proposition 6.* The proof relies on showing that for any finite number of iterations of operator $B^\Delta$ it is possible to get an arbitrarily close approximation of $V(0)$, when the initial set in the iteration consists of $v(p, \Delta)$ as in Lemma 8 and $\Delta$ is small enough.

Let us set $W^0(\Delta) = \{v(p, \Delta)\}$, where $v(p, \Delta)$ is as in Lemma 8. The set $W^{k,\Delta}(\Delta')$ denotes the set obtained in the $k$'th step of the iteration (4) for $B^\Delta$, when the iteration is started with $W^0(\Delta')$. The set $W^k(0)$ denotes the set obtained in the $k$'th iteration of (A.1) for $B^0$, when the iteration is started with $\{v\}$, where $v$ is the limit of $v(p,\Delta)$, when $\Delta$ goes to zero.

First, note that $W^{k,\Delta}(\Delta) \subseteq V(\Delta)$ and $W^k(0) \subseteq V(0)$, because $W^0(\Delta)$ are equilibrium payoffs. Now consider the Hausdorff distance between $V(\Delta)$ and $V(0)$. It follows from the triangular inequality that

$$\begin{aligned} d_{\mathcal{H}}\left(V(\Delta),V(0)\right) \leq & d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta),V(0)\right) \\ \leq & d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta),W^k(0)\right) + d_{\mathcal{H}}\left(W^k(0),V(0)\right). \end{aligned}$$

Applying the triangular inequality to the term $d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta),W^k(0)\right)$ gives

$$d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta),W^k(0)\right) \leq d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta),W^{k,\Delta}(0)\right) + d_{\mathcal{H}}\left(W^{k,\Delta}(0),W^k(0)\right).$$

Altogether, it holds that

$$\begin{aligned} d_{\mathcal{H}}\left(V(\Delta),V(0)\right) \leq & d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta),W^{k,\Delta}(0)\right) + d_{\mathcal{H}}\left(W^{k,\Delta}(0),W^k(0)\right) \\ & + d_{\mathcal{H}}\left(W^k(0),V(0)\right). \end{aligned} \tag{A.2}$$

Let us next consider the first term on the right hand side of (A.2). Because the mappings $(I-e^{-r\Delta})u(a)+e^{-r\Delta}v$, $a \in A$, are continuous as functions of $v$, by choosing $\Delta$ small enough the trajectories $\phi^{\Delta}_{av}$ and $\phi^{\Delta}_{av^1}$ for $v^1 = v(p,\Delta)$, become arbitrarily close to each other. Notice that some of the points on $\phi^{\Delta}_{av}$ and $\phi^{\Delta}_{av^1}$ may not be incentive compatible if $u_i(a) < v_i^-$ for $i \in N' \subseteq N$. However, choosing $\Delta$ small enough both of these trajectories have some incentive compatible points arbitrarily close to the set

$$\left\{v \in FP : (1-e^{-r_i\Delta})u_i(a) + e^{-r_i\Delta}v_i = (1-e^{-r_i\Delta})d_i(a) + e^{-r_i\Delta}v_i^-,\ i \in N'\right\}.$$

Hence, it follows that $W^{1,\Delta}(0) = B^{\Delta}(v)$ and $W^{1,\Delta}(\Delta) = B^{\Delta}(v^1)$ can be made arbitrarily close to each other by choosing $\Delta$ small enough. The argument can be repeated for any pair of points in $W^{1,\Delta}(0)$ and $W^{1,\Delta}(\Delta)$, which implies that $W^{2,\Delta}(0)$ and $W^{2,\Delta}(\Delta)$ become arbitrarily close to each other when $\Delta$ is small enough. By repeating the argument, the result holds for $W^{k,\Delta}(0)$ and $W^{k,\Delta}(\Delta)$ for any $k$. Hence, the first term in (A.2) can be made arbitrarily small for any $k$.

Let us next turn to the second term on the right hand side of (A.2). Recall that the distance between two consecutive points in a trajectory is at most $\max_i(1-e^{r_i\Delta})\max_i|u_i(a)-v_i|$, see Remark 1. Because the term $\max_i|u_i(a)-v_i|$ is bounded, the distance is bounded by a function $\rho(\Delta)$ for any $a$ and $v$, and this function $\rho(\Delta)$ is proportional to $\max_i(1 - e^{-r_i\Delta})$. The distance between the orbit $\phi^0_{av}$ an the trajectory $\phi^\Delta_{av}$ is at most the same as the distance between two consecutive points in the trajectory. It follows that the Hausdorff distance between $B^0(W)$ and $B^\Delta(W)$ is bounded by $\rho(\Delta)$ for any compact set $W$. Hence, $d_{\mathcal{H}}\left(W^{k,\Delta}(0), W^\Delta(0)\right)$ can be made arbitrarily small by choosing $\Delta$ small enough.

Finally, let us consider the third term on the right hand side of (A.2). The convergence of the fixed-point iteration (Lemma 6) implies that for any $\varepsilon > 0$ it is possible to choose $k$ such that

$$d_{\mathcal{H}}\left(W^k(0), V(0)\right) < \varepsilon/3,$$

and, as argued above, corresponding to this $k$ it is possible to choose $\Delta_\varepsilon$ small enough such that

$$d_{\mathcal{H}}\left(W^{k,\Delta}(\Delta), W^{k,\Delta}(0)\right), d_{\mathcal{H}}\left(W^{k,\Delta}(0), W^k(0)\right) < \varepsilon/3.$$

for all $0 < \Delta < \Delta_\varepsilon$. The inequality (A.2) implies that for any $\varepsilon > 0$ there is $\Delta_\varepsilon > 0$ such that $d_{\mathcal{H}}(V(\Delta), V(0)) < \varepsilon$ for all $0 < \Delta < \Delta_\varepsilon$. □

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Contact information: Aboa Centre for Economics, Department of Economics, Rehtorinpellonkatu 3, FI-20500 Turku, Finland.

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