Hannu Salonen

Bonacich Measures as Equilibria in Network Models

Aboa Centre for Economics
Discussion paper No. 100
Turku 2015

The Aboa Centre for Economics is a joint initiative of the economics departments of the University of Turku and Åbo Akademi University.

ISSN 1796-3133

Printed in Uniprint
Turku 2015

Hannu Salonen

Bonacich Measures as Equilibria in Network Models

Aboa Centre for Economics
Discussion paper No. 100
June 2015

ABSTRACT

We investigate the cases when the Bonacich measures of strongly connected directed bipartite networks can be interpreted as a Nash equilibrium of a non-cooperative game. One such case is a two-person game such that the utility functions are bilinear, the matrices of these bilinear forms represent the network, and strategies have norm at most one. Another example is a two-person game with quadratic utility functions. A third example is an $m + n$ person game with quadratic utilitity functions, where the matrices representing the network have dimension $m \times n$. For connected directed bipartite networks we show that the Bonacich measures are unique and give a recursion formula for the computation of the measures. The Bonacich measures of such networks can be interpreted as a subgame perfect equilibrium path of an extensive form game with almost perfect information.

JEL Classification: C71, D85

Keywords: networks, influence measures, Nash equilibrium

Contact information

Hannu Salonen
Department of Economics
University of Turku
FI-20014, Finland
Email: hannu.salonen (at) utu.fi

Acknowledgements

I am grateful to Mitri Kitti and Matti Pihlava for useful comments and discussions. I thank the Academy of Finland and the OP-Pohjola Group Research Foundation for financial support.

1. Introduction

We investigate the cases when the Bonacich measures of strongly connected directed bipartite networks can be interpreted as a Nash equilibrium of a non-cooperative game. One such case is a two-person game such that the utility functions are bilinear, the matrices of these bilinear forms represent the network, and strategies have norm at most one. Another example is a two-person game with quadratic utility functions. A third example is an $m+n$ person game with quadratic utilitity functions, where the matrices representing the network have dimension $m \times n$. For connected directed bipartite networks we show that the Bonacich measures are unique and give a recursion formula for the computation of the measures. The Bonacich measures of such networks can be interpreted as a subgame perfect equilibrium path of an extensive form game with almost perfect information.

In social networks *nodes* are agents who are connected to other agents via *links*. Links could have a weight that indicates how important the connection is. Links could also be directed if two agents are not symmetrically related to each other.

In network theory so called centrality measures attempt to quantify the importance or influence of an agent in a social network. Centrality measures resemble solution concepts of cooperative game theory. The Shapley value, for example, has been applied in both theories (Shapley 1953).

There is large literature about connections between solutions concepts of cooperative game theory and equilibrium concepts of noncooperative game theory. The "Nash programme" was initiated already by John Nash, who thought that noncooperative game theory is more fundamental than the cooperative theory (Nash 1950, Nash 1953). In this view, "reasonable solutions" of cooperative games should be equilibrium outcomes of some (equally reasonable) noncooperative game (see also Binmore 1987, Rubinstein 1982).

In network theory there are now some papers where centrality measures are shown to be equilibria of noncooperative games. Representative papers are Ballester *et.al* (2006) and Cabrales *et.al* (2011), who show that the Katz-Bonacich centrality measure is a Nash equilibrium (see also Katz 1953, Bonacich 1987, Bonacich and Lloyd 2001, Jackson and Zenou 2014). In their models agents choose a real number that indicates how much an agent invests in network activities. An adjacecncy matrix gives the link structure of the network, and payoff functions are quadratic.

Salonen (2014) studies noncooperative link formation games in which agents decide how much to invest in relations with every other agent. Strategy sets are standard simplices and payoff functions are of Cobb-Douglas type. Players have a common popularity ordering over their opponents: it tells how valuable it is to be in contact with other players.

The equilibrium strategies are computed as well as the values of some standard centrality measures from the resulting equilibrium networks. The measures analyzed are the eigenvector centrality, indegree, the Katz-Bonacich measure, and the PageRank measure (see Bonacich 1972, Brin and Page 1998, Freeman 1979). Depending on assumptions of the parameters of payoff functions, the centrality measures give the same or the opposite ordering of the players as the common popularity ordering.

To my knowledge the present paper is the first one in which equilibria and centrality measures of bipartite directed networks are compared. In a bipartite network the node set consist of two separate sets of nodes, say V_1 and V_2, and there cannot be any links within V_1 or V_2, but all links are between these groups.

Bonacich (1991) initiated the study of such networks in social sciences. In his paper, networks are unweighted and undirected and in this case each connected bipartite network has two Bonacich measures. These measures are defined in the same way for weighted undirected bipartite networks. However, if a bipartite network is directed and *strongly connected*, then the network has *four* Bonacich measures, although they are defined in the similar manner as in Bonacich (1991).

The paper is organized as follows. In Section 2 basic definitions and notation are given. In subsection 2.1 we take a closer look at the bipartite networks, Bonacich measures, and their potential applications. In Section 3 main results are stated and proven.

2. Preliminaries

A *network* $G = (V, E)$ consists of a finite set V of *nodes*, and a finite set E of *links* between nodes. A network is *directed*, if links are represented by a set of ordered pairs $E = \{(i, j) \mid i, j \in V\}$. We may also denote $i \to j$ if there is a link from i to j.

A network G can be represented by a matrix A whose rows and columns are indexed by the nodes $i \in V$. A network is *weighted*, if entries a_{ij} can take any nonnegative values, representing the *strength* or intensity of the link from i to j. If $a_{ij} = 0$ then there is no link from i to j. In this paper the networks will be directed and weighted.

A network is *bipartite* (sometimes called a *bimodal* or an *affiliation* network), if the node set V is partitioned into two nonempty subsets V_1 and V_2 such that if two nodes are connected then they cannot be elements of the same partition member V_i. A bipartite network can be represented by an $m \times n$ matrix A, where $m = |V_1|$ and $n = |V_2|$.

A weighted directed bipartite network G may be represented by a pair (A, B), where A and B are both $m \times n$ nonnegative matrices. An entry a_{ij} of A would give the weight of the link from $i \in V_1$ to $j \in V_2$, and b_{ij} would give the the weight of the link from $j \in V_2$ to $i \in V_1$.

A *path* from a node $i \in V$ to node $j \in V$ is a subset of nodes $i_0, \ldots, i_k$ such that $i = i_0, i_k = j$, $i_j \to i_{j+1}$ for all $j = 0, \ldots, k-1$, and $i_n \neq i_m$ for all $n, m < k$, $n \neq m$. A path is a *cycle*, if $i = j$.

A subset C of nodes V of a directed network is *connected*, if for every $i, j \in C$ there is a path from i to j or a path from j to i. A connected subset C is *strongly connected*, if for every $i, j \in C$ there is a path from i to j and a path from j to i.

A (strongly) connected subset C is a (strongly) connected *component* if no proper superset of C is (strongly) connected. That is, C is a maximal (strongly) connected subset of V.

If any node $i \in V$ of a directed network belongs to a strongly connected component, then the node set V can be partitioned into disjoint strongly connected components. If there are some nodes that do not belong to any strongly connected component, then V has a partition such that partition members are either strongly connected components or singletons.

A collection $\mathcal{C}$ of strongly connected components is connected, if the members of the collection can be indexed by numbers i such that $\mathcal{C} = \{V^1, \ldots, V^t\}$, and there is a link from V^j to V^{j-1}, $j = 2, \ldots, t$. Note that there cannot be any links from V^k to V^j, if $j > k$. Therefore we may say that $\{V^1, \ldots, V^t\}$ is an *ordered collection.*

The node set V of a directed network can not always be partitioned into connected components.

Example 1. Let $V = \{1, 2, 3, 4, 5, 6\}$, and let $\{1, 2\}$, $\{3, 4\}$, and $\{5, 6\}$ be the strongly connected components of a directed network. Assume that there are only two other links: $5 \to 1$ and $6 \to 3$. Then $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$ are the only connected components, but they do not partition V.

2.1. Bonacich measures and applications

Let $G = (V_1 \cup V_2, E)$ be a weighted directed bipartite network represented by a pair (A, B) of $m \times n$ matrices. We may call rows of the matrices *agents* who are members of some *clubs* that are represented by the columns. For example, a_{ij} could be the value agent i gets from club j, and b_{ij} could measure the contribution agent i makes to club j.

A typical entry a_{ij} of A (b_{ij} of B) gives the strength of the link from agent i to club j (from club j to agent i). If $a_{ij} = 0$ ($b_{ij} = 0$), we interpret that there is no link from i to j (from j to i).

For example, rows could be researchers and and columns could be journals. Value a_{ij} of a link from $i \in V_1$ to $j \in V_2$ could give the number of times author i cites an article in journal j. Value b_{ij} could be number of times author i's paper has been cited in journal j

As another potential application, let V_1 and V_2 be subsets of teachers and students, respectively, of some university. Entry a_{ij} could be the grade (quality of teaching) student i gives to teacher j, and b_{ij} could be the grade (of an exam) teacher j gives to student i.

Given a pair (A, B) of matrices representing a bipartite network with a node set $V_1 \cup V_2$, we can form unimodal networks G_1 and G_2 with node sets V_1 and V_2 as follows.

Consider the product matrix AB^T. A typical entry $(AB^T)_{ij}$ of AB^T (an $m \times m$ matrix) is of the form $\sum_k a_{ik}b_{kj}$: it is the sum of products $a_{ik}b_{kj}$ over k, where this product gives the strength of the directed path $i \to k \to j$ from agent i to agent j via club k. We will denote by G_1 the directed network of agents (nodes in V_1) represented by AB^T.

Product matrix $B^T A$ (an $n \times n$ matrix) would give directed network between clubs. Typical entry $(B^T A)_{ij}$ is of the form $\sum_k b_{ki}a_{kj}$, where $b_{ki}a_{kj}$ is the strength of the path $i \to k \to j$ from club i to club j via agent k. We will denote by G_2 the directed network of clubs (nodes in V_2) represented by $B^T A$.

The left and right eigenvectors $y^T B^T A = \lambda y^T$ and $B^T Ax = \lambda x$ (corresponding to the largest eigenvector λ of $B^T A$) measure different things. The left eigenvector y_i measures the importance of club i in terms of outgoing links, and x_i measures the importance of club i in terms of incoming links. Let's put these into equation

$$y^T B^T A = \lambda y^T \tag{1}$$

$$B^T Ax = \lambda x. \tag{2}$$

The eigenvectors y and x are called the *Bonacich measures* of a directed network G_2 represented by $B^T A$. Bonacich (1991) originally defined these measures for unweighted undirected bipartite networks. Kleinberg (1999) has proposed a measure called HITS for unweighted directed networks that can be represented by square matrices. HITS is calculated in the same way as the Bonacich measure.

We may sometimes call y the (Bonacich) row measure, and x the (Bonacich) column measure. Bonacich measures for the network G_1 are similarly defined as the eigenvectors of AB^T. Note that the left (right) eigenvectors of AB^T are the right (left) eigenvectors of BA^T, the transpose of AB^T.

By the Perron-Frobenius theorem the eigenvectors of B^TA are unique (up to multiplication by positive constants), if the nonnegative square matrix B^TA is *irreducible*. That is, for any i, j there exists a strictly positive integer $t(i,j)$ such that the (i,j) entry of $(B^TA)^{t(i,j)}$ is strictly positive, where X^t denotes the t'th power of matrix X. This means that the directed network G_2 represented by B^TA is strongly connected.

Note that B^TA may be irreducible although AB^T is not irreducible. But if both of these matrices are irreducible, then a directed bipartite network G that is represented by (A, B) is strongly connected. In this case G has *four* Bonacich measures associated with it, since both B^TA and AB^T have two eigenvectors each.

We give next another way of getting all four Bonacich measures of a directed bipartite network (this method was used already by Bonacich 1991). Consider the system of equations

$$Bp = \alpha q \tag{3}$$

$$q^TA = \beta p^T. \tag{4}$$

Now $Bp = \alpha q$ means that $p^TB^T = \alpha q^T$. Hence $q^T = (1/\alpha)p^TB^T$, and therefore from equation (4) we get that $(1/\alpha)p^TB^TA = \beta p^T$. On the other hand, form equation (4) we get that $p = (1/\beta)A^Tq$, and then equation (3) implies $BA^Tq = \alpha\beta q$. We have the following equations

$$p^TB^TA = \alpha\beta p^T \tag{5}$$

$$BA^Tq = \alpha\beta q. \tag{6}$$

If the networks G_1 and G_2 are strongly connected, the left eigenvectors y and p of B^TA in equations (1) and (5) are the same.

As an example, row i of A could consist of the grades student i *gives* to schools, and column j of B could consist of the grades school j *gives* to students. Then vector p consists of the weights of schools as evaluators, and vector q consists of the weights of students as evaluators. Student i gets a high grade as an evaluator, if he has given a high score to those schools that have received high scores from other students on the average.

If we reformulate equations (3) and (4) and look solutions of

$$z^T B = \alpha' w^T \tag{7}$$

$$Aw = \beta' z \tag{8}$$

we get that

$$z^T BA^T = \alpha' \beta' z^T \tag{9}$$

$$B^T Aw = \alpha' \beta' w \tag{10}$$

If both G_1 and G_2 are strongly connected, then the following relations hold. The vector z in equation (9) is the left eigenvector of BA^T, and q in equation (6) is it's right eigenvector. The right eigenvector w of $B^T A$ in equation (10) is the same as x in equation (2). The left eigenvector p of $B^T A$ is the same as y in equation (1).

By irreducibility of BA^T and $B^T A$, the eigenvalues λ of equations (1) and (2), $\alpha\beta$ of equations (5) and (6), and $\alpha'\beta'$ of equations (9) and (10) are the same. (It holds in general that if X is an $m \times n$ matrix and Y is an $n \times m$ matrix, then the non-zero eigenvalues of XY and YX are the same.)

We can interpret equations (7) and (8) in the example where students and teachers are grading each other. In this case, row i of A consists of the grades student i *gets* from schools, and column j of B consists of the grades school j *gets* from the students. Vector z gives the values of students as their "weighted grade point averages" from schools, and vector w is a similar measure for schools.

So if we look at the eigenvectors y and x of $B^T A$ in equations (1) and (2), we note that the left eigenvector $y = p$ gives a measure of schools as evaluators, and the right eigenvector $x = w$ measures the quality of schools in the eyes of students. The left and right eigenvectors q and z of AB^T are corresponding measures of students.

3. Results

Throughout this section (A, B) is a pair of nonnegative $m \times n$ matrices representing a directed bipartite network G. We assume also *w.l.o.g.* that each node $v \in V_1 \cup V_2$ has at least one link.

Let $\|\cdot\|$ denote the Euclidean norm. Let $B^k = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ be the unit ball of $\mathbb{R}^k$, and let $B^k_+ = B^k \cap \mathbb{R}^k_+$ be the intersection of the unit ball with the nonnegative orthant of $\mathbb{R}^k$.

Given a directed bipartite network G, let G^* denote the following two-person normal form game associated with G. The strategy set of player 1 ("row player") is $S_1 = B^m_+$, and the strategy set of player 2 ('column player") is $S_2 = B^n_+$. The utility function of player 1 is $u_1(x, y) = x^T Ay$, and the utility function of player 2 is $u_2(x, y) = x^T By$, where $x \in S_1, y \in S_2$.

Proposition 1. *Let (A, B) be a pair of nonnegative $m \times n$ matrices representing a strongly connected directed bipartite network G. Let G^* be the two-person normal form game associated with G. If $z \in S_1$ and $w \in S_2$ are the eigenvectors given by equations (9) and (10) such that $\|z\| = \|w\| = 1$, then (z, w) is a Nash equilibrium of G^*. If $(z, w) \neq (0, 0)$ is a Nash equilibrium of G^*, then z and w are the eigenvectors given by equations (9) and (10) such that $\|z\| = \|w\| = 1$.*

Proof. Let $(z, w) \in S_1 \times S_2$ satisfy equations (9) and (10), and assume $\|z\| = \|w\| = 1$. Equation (8) implies that $u_1(x, w) = \beta' x^T z$. Hence the best reply for player 1 against w is z. Similarly, equation (7) implies that $u_2(z, y) = \alpha' w^T y$. Hence the best reply for player 2 against z is w. Therefore (z, w) is a Nash equilibrium of G^*.

Let $(z, w) \neq (0, 0)$ be a Nash equilibrium of G^*. The first order condition for $z_i > 0$ of player 1 is

$$\sum_j a_{ij} w_j = \frac{\beta' z_i}{\sqrt{\sum_k z_k^2}},$$

where β' is the Lagrangian multiplier of the constraint $\|z\| \leq 1$. Now $z_i > 0$ implies $\beta' > 0$ and hence $\|z\| = \sqrt{\sum_k z_k^2} = 1$. Therefore the first order conditions corresponding to $z_i > 0$ of player 1 satisfy the equations

$$Aw = \beta' z.$$

By the same reasoning, the first order conditions corresponding to $w_j > 0$ of player 2 satisfy

$$z^T B = \alpha' w^T.$$

But these first order conditions are the same as equations (7) and (8), and we are done. □

The strategy spaces of the players in game G^* are nonnegative vectors having norm at most one. Let us see what kind of relations there are between Nash equilibria and Bonacich measures if the strategy sets are standard simplices.

Denote by G^{Δ} a two-person normal form game such that the strategy set for players 1 and 2 are $\Delta^m = \{x \in \mathbb{R}_+^m \mid \sum_i x_i = 1\}$ and $\Delta^n = \{y \in \mathbb{R}_+^n \mid \sum_i y_i = 1\}$, respectively. The utility functions of players are $u_1(x, y) = x^T Ay$ and $u_2(x, y) = x^T By$.

Proposition 2. *Let z, w be the Bonacich measures of a strongly connected directed bipartite network G represented by (A, B). Let G^{Δ} the two-person normal form game associated with G. Then $(z, w) \neq (0, 0)$ is a Nash equilibrium of G^{Δ}, iff z and w are uniform distributions.*

Proof. First note that we can normalize Bonacich measures z and w so that $z \in \Delta^m$ and $w \in \Delta^n$ (that may change eigenvalues but not eigenvector spaces). Since G is strongly connected, both z and w are strictly positive.

Strategy z is a best reply against w, if and only if all elements of the vector Aw are the same. But then $z = (1/m, \dots, 1/m)$ by equation (8). By the same argument $w = (1/n, \dots, 1/n)$ □

Example 2. Let A and B be the following row and column stochastic matrices, respectively

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Then the matrices AB^T and B^TA are the following row stochastic matrices

$$AB^T = \frac{1}{9}\begin{bmatrix} 2 & 3 & 4 \\ 2 & 2 & 5 \\ 3 & 2 & 4 \end{bmatrix}, \quad B^TA = \frac{1}{9}\begin{bmatrix} 4 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 5 & 1 \end{bmatrix}$$

The right eigenvector of both AB^T and B^TA is $(1/3, 1/3, 1/3)$. The left eigenvector of AB^T is $(24/91, 23/91, 43/91)$, and the left eigenvector of B^TA is $(38/91, 31/91, 22/91)$.

As a third game consider a two-person game G^c such that player 1's strategies are vectors $x \in \mathbb{R}^m_+$ and player 2's strategies are vectors $y \in \mathbb{R}^n_+$. Player 1 has utility function $u_1(x, y) = x^TAy - (c_1/2)(\sum_i x_i^2)$ and player 2 has utility function $u_2(x, y) = x^TBy - (c_2/2)(\sum_j y_j^2)$, where $c_1, c_2 > 0$ are constants.

Proposition 3. *Let G be a strongly connected directed bipartite network represented by (A, B). A strategy pair $(x, y) \neq (0, 0)$ is a Nash equilibrium of G^c, iff x and y are the Bonacich measures of G.*

Proof. Given $x' \in \mathbb{R}^m_+$ and $y' \in \mathbb{R}^n_+$, the best replies x and y for players 1 and 2 satisfy the first order conditions

$$Ay' = c_1 x$$
$$x'^TB = c_2 y^T.$$

These equations resemble equations (7) and (8). A vector (x, y) is a Nash equilibrium iff $x = x'$ and $y = y'$. Multiplying the first equation by β'/c_1 and the second equation by α'/c_2 does not change the equilibria. But then these equations are like equations (7) and (8), except that the matrices are the same up to a positive constant. But such a change in matrices does not change the eigenvector spaces although eigenvalues may change. Hence the Nash equilibria are also Bonacich measures.

The proof to the other direction follows the same logic. □

Finally, let us study a $m + n$ player game $G^{m,n}$ defined as follows. The player set is $\{1, \ldots, m, m + 1, \ldots, m + n\}$. The strategy set of each player i is $\mathbb{R}_+$. Let us denote by $x = (x_1, \ldots, x_m)$ the strategy profile of the first m players and by $y = (y_{m+1}, \ldots, y_{m+n})$ the strategy profile of the remaining players. The utility function of a player $i \leq m$ is $u_i(x_i, x_{-i}, y) = (x_i, x_{-i})^T Ay - (c_1/2)x_i^2$, where $c_1 > 0$. The utility function of a player $i > m$ is $u_i(x, y_i, y_{-i}) = x^T B(y_i, y_{-i}) - (c_2/2)y_i^2$, where $c_2 > 0$.

Proposition 4. *Let G be a strongly connected directed bipartite network represented by (A, B). A strategy profile $(x, y) \neq (0, 0)$ is a Nash equilibrium of $G^{m,n}$, iff x and y are the Bonacich measures of G.*

Proof. The first order conditions of players $i \leq m$ and $j > m$ are

$$\sum_k a_{ik} y_k = c_1 x_i, \;\; i = 1, \ldots, m$$
$$\sum_t b_{tj} x_t = c_2 y_j, \;\; j = m + 1, \ldots, m + n.$$

The rest of the proof is similar to the proof of Proposition 3. □

A possible interpretation of Proposition 4 is that the choice variable x_i of "row agent" i gives the amount of resources agent i invests in network activities. The i'th row of A says how much utility a_{ij} each unit invested returns from activity j. Column agents' choices y_j and utilities are interpreted in the same way.

3.1. A recursive construction of Bonacich measures for connected networks

Suppose (A, B) represents a *connected* directed bipartite network with a node set $V = V_1 \cup V_2$ such that any node is a member of a strongly connected component. Let $\mathcal{C} = \{V^1, \ldots, V^k\}$ be the partition of V into strongly connected components. Since $\mathcal{C}$ necessarily is an ordered collection we may choose the indices $i \in \{1, \ldots, k\}$ in such a way that for all $j < k$ there exist $v \in V^j$ and $v' \in V^{j+1}$ such that $v' \to v$. Note that if $j < t$, there cannot be any links from V^j to V^t, since V^j and V^t are strongly connected components.

Each $V^i \in \mathcal{C}$ is of the form $V^i = V_1^i \cup V_2^i$, where $V_1^i \subset V_1$ and $V_2^i \subset V_2$. Since by assumption each V^i contains at least two nodes, V_1^i and V_2^i are nonempty subsets for all i. We will index nodes in such a way that each V_1^i consists of consecutive natural numbers from the set $\{1, \ldots, m\}$, so that $V_1^1 = \{1, \ldots, m_1\}$, $V_1^2 = \{m_1 + 1, \ldots, m_2\}$, and so on. Nodes in V_2 are indexed analogously.

We will construct Bonacich row measure z and column measure w for the network G by starting from the first component V^1, and solving recursively the measures for components V^j, $j > 1$.

The first member of the collection $\mathcal{C}$ is V^1, and V^1 has no links that extend out of V^1. That is, if $v \in V^1$ and $v \to y$, then $y \in V^1$. The nodes in V_1^1 correspond to the rows $1, \ldots, m_1$ of matrices A and B, and nodes in V_2^1 correspond to the columns $1, \ldots, n_1$ of matrices A and B. Let $A(1)$ and $B(1)$ be the matrices corresponding to these blocks of A and B, respectively.

For any $i \in V_1^1$ (any $j \in V_2^1$), the entry a_{ij} (b_{ij}) of A (B) is strictly positive only if $j \leq n_1$ ($i \leq m_1$). On the other hand, the matrices $A(1)B^T(1)$ and $B^T(1)A(1)$ are irreducible. Hence the left and right eigenvectors for these matrices exist uniquely. So there exist unique $z(1)$ and $w(1)$ satisfying equations (9) and (10), where these vectors have the dimensions m_1 and n_1, respectively. The coordinates of $z(1)$ and $w(1)$ are all strictly positive, and $\|z(1)\| = \|w(1)\| = 1$ *w.l.o.g.*

Let the rows $m_1 + 1, \ldots, m_2$ correspond to the nodes in V_1^2, and let the columns $n_1+1,\ldots,n_2$ correspond to the nodes in V_2^2. Let $A(2)$ be the matrix consisting of the first m_2 rows of A and the first n_2 columns of A. Let $B(2)$ be the matrix consisting of the first m_2 rows of B and the first n_2 columns of B.

Consider the following equations, analogous to equations (7) and (8):

$$z^T B(2) = \alpha w^T \tag{11}$$

$$A(2)w = \beta z \tag{12}$$

where z is a nonnegative m_2 dimensional vector who's first m_1 coordinates are given by the vector $z(1)$, and w is a nonnegative n_2 dimensional vector who's first n_1 coordinates are given by the vector $w(1)$.

Note that both $A(2)B^T(2)$ and $B^T(2)A(2)$ cannot be irreducible, since the node set $V^1 \cup V^2$ is not strongly connected. However, there exists unique vectors $z(2)$ and $w(2)$ satisfying these equations, for some $\alpha, \beta > 0$. Let us prove this claim next.

First note the matrices $A(2)$ and $B(2)$ have the following block form

$$A(2) = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}, \; B(2) = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix}$$

Pairs $(A^{11}, B^{11}) = (A(1), B(1))$ and (A^{22}, B^{22}) correspond to the strongly connecetd components V^1 and V^2

For any $i \in V_1^2$, all the entries a_{ij} such that $i \leq m_1$ and $j > n_1$ are zero. For any $j \in V_2^2$, all the entries b_{ij} of B such that $i > m_1$ and $j \leq n_1$ are zero. These constraints hold because there cannot be a link from v to v' if $v \in V^1$ and $v' \in V^2$. Therefore A^{12} and B^{21} are null matrices.

Equations (11) and (12) can then be rewritten as

$$z(1)^T B^{12} + z(2)^T B^{22} = \beta w(2)^T \tag{13}$$

$$A^{21} w(1) + A^{22} w(2) = \alpha z(2) \tag{14}$$

where $z(2)$ has dimension $m_2 - m_1$ and $z(2)$ has dimension $n_2 - n_1$. Vectors $z(1)$ and $w(1)$ are the normalised Bonacich measures of the strongly connected component V^1 satisfying equations (9) and 10.

Solve for $w(2)$ from equation (13) and insert it into equation (14). Solve also for $z(2)$ from equation (14) and insert it into equation (13). We get the following

$$A^{22}\left(B^{22}\right)^T z(2) = \alpha\beta z(2) - \beta A^{21} w(1) - A^{22}\left(B^{12}\right)^T z(1) \tag{15}$$

$$w(2)^T\left(A^{22}\right)^T B^{22} = \alpha\beta w(2)^T - \alpha z(1)^T B^{12} - w(1)^T\left(A^{21}\right)^T B^{22} \tag{16}$$

The following Lemma states that these equations have a unique solution $z(2), w(2)$ and that these vectors are strictly positive.

Notation. Given two vectors $x = (x_1, \ldots, x_t)$ and $y = (y_{t+1}, \ldots, y_m)$, we denote by $x^\frown y$ the concatenation of x and y: $x^\frown y = (x_1, \ldots, x_t, y_{t+1}, \ldots, y_m)$. Concatenation $x(1)^\frown \cdots ^\frown x(k)$ over several vectors $x(1), \ldots, x(k)$ is defined analogously. We may denote by $\vec{x}(k)$ the concatenation of vectors $x(1), \ldots, x(k)$: $\vec{x}(k) = x(1)^\frown \cdots ^\frown x(k)$.

Lemma 1. *Let $z(1)$ and $w(1)$ be the unique eigenvectors having norm 1 and satisfying equations (9) and (10) for the pair (A^{11}, B^{11}) representing a strongly connected network on a node set V^1. Then equations (15) and (16) have unique solutions $z(2), w(2)$, and these vectors are strictly positive.*

Proof. There exists at least one solution $z(2), w(2)$ to equations (15) and (16), for some $\alpha, \beta > 0$. To see this, add $\varepsilon > 0$ to the entries of matrices $A(2)$ and $B(2)$ in equations (11) and 12. Then these matrices would correspond a strongly connected bipartite network with node set $V^1 \cup V^2$. Using these matrices in equations (11) and (12), strictly positive solutions $z^\varepsilon, w^\varepsilon$ will exist. These vectors can be chosen to have norm 1. With this normalization, z^ε and w^ε are unique solutions, for any given $\varepsilon > 0$.

Letting ε approach zero, all cluster points of $\{(z^\varepsilon, w^\varepsilon)\}_{\varepsilon\downarrow 0}$ are nonnegative, and they are solutions of the original equations (11) and (12). Such cluster points exist because the subset of vectors z, w having norm 1 is a compact set. Note also that while α and β may depend on ε, all values of α and β stay in a bounded set because z^ε and w^ε have norm 1.

Equations (15) and (16) must hold approximatively for z^ε and w^ε when ε is small, and the approximation must get better and better as ε goes to 0. This holds since matrices A^{12} and B^{21} approach null matrices as ε goes to 0

Let (z^*, w^*) be an arbitrary cluster point of $\{(z^\varepsilon, w^\varepsilon)\}_{\varepsilon\downarrow 0}$. Note that (z^*, w^*) solves also equations (15) and (16), because the first m_1 and n_1 coordinates of z^* and w^* must be proportional to vectors $z(1)$ and $w(1)$. Further, since values of α and β stay in a bounded set, at least some coordinates z_i^* and w_j^*, for $i > m_1, j > n_1$, must be strictly positive by equations (15) and (16). We show now that z^* and w^* are both strictly positive.

From equations (15) and (16) we get the following vector inequalities:

$$A^{22}\left(B^{22}\right)^T z^*(2) \le \alpha\beta z^*(2) \tag{17}$$

$$w^*(2)^T\left(A^{22}\right)^T B^{22} \le \alpha\beta w^*(2)^T \tag{18}$$

where $z^*(2) = (z^*_{m_1+1}, \ldots, z^*_{m_2})$ and $w^*(2) = (w^*_{n_1+1}, \ldots, w^*_{n_2})$. From inequality (17) we get that there exists $\alpha'\beta' > 0$ such that (17) holds as equality. But then $z^*(2)$ is the eigenvector corresponding to the largest eigenvalue $\alpha'\beta' > 0$ of $A^{22}\left(B^{22}\right)^T$, and hence $z^*(2)$ is strictly positive. Then $w^*(2)$ in equation (18) must be the left eigenvector of $\left(A^{22}\right)^T B^{22}$ corresponding to $\alpha'\beta' > 0$, because $z^*(2), w^*(2)$ solves equations (9) and (10). Hence $w^*(2)$ must also be strictly positive.

But then (z^*, w^*) is the unique cluster point of $\{(z^\varepsilon, w^\varepsilon)\}_{\varepsilon\downarrow 0}$. Choose $a >$ and $b > 0$ such that $a(z_1^*, \ldots, z^*_{m_1}) = z(1)$ and $b(w_1^*, \ldots, z^*_{n_1}) = w(1)$, and let $z(2) = a(z^*_{m_1+1}, \ldots, z^*_{m_2})$ and $w(2) = b(w^*_{n_1+1}, \ldots, w^*_{n_2})$. Then $z(1)^\frown z(2)$ and $w(1)^\frown w(2)$ are Bonacich measures of the connected network with node set $V^1 \cup V^2$, because they are the unique solutions of equations (11) and (12). $\square$

We can use equations (15) and (16) to solve $w(2)$ and $z(2)$ as functions of $w(1)$ and $z(1)$.

$$z(2) = \frac{\beta_2 A^{21} w(1) + A^{22} \big(B^{12}\big)^T z(1)}{\alpha_2 \beta_2 - \alpha_2' \beta_2'} \tag{19}$$

$$w(2) = \frac{\alpha_2 \big(B^{12}\big)^T z(1) + \big(B^{22}\big)^T A^{21} w(1)}{\alpha_2 \beta_2 - \alpha_2' \beta_2'} \tag{20}$$

where $A^{22}\big(B^{22}\big)^T z(2) = \alpha_2' \beta_2' z(2)$ and $\big(B^{22}\big)^T A^{22} w(2) = \alpha_2' \beta_2' w(2)$.

The eigenvalues obtained at the first and second step of recursion are different. To make this point explicit we have indicated the eigenvalues obtained at the second step by subscripts: α_2, β_2. Note also that $\alpha_2 \beta_2 > \alpha_2' \beta_2'$.

We can apply one-to-one the methods used in in the proof of Lemma 1 to construct Bonacich measures for the connected bipartite network with a node set $V = V^1 \cup \cdots \cup V^k$. Suppose Bonacich measures $\vec{z}(m-1) = z(1)^\frown \cdots ^\frown z(m-1)$ and $\vec{w}(m-1) = w(1)^\frown \cdots ^\frown w(m-1)$ of the connected network with node set $V^1 \cup \cdots \cup V^{m-1}$ have been found, $1 < m \leq k$.

Replace $A(1)$ and $B(1)$ by the matrices $A(m-1)$ and $B(m-1)$ whose rows correspond to nodes in $V_1^1 \cup \cdots \cup V_1^{m-1}$ and columns correspond to nodes in $V_2^1 \cup \cdots \cup V_2^{m-1}$. Replace the Bonacich measures $z(1)$ and $w(1)$ by the Bonacich measures $\vec{z}(m-1) = z(1)^\frown \cdots ^\frown z(m-1)$ and $\vec{w}(m-1) = w(1)^\frown \cdots ^\frown w(m-1)$. Replace $A(2)$ and $B(2)$ by the blocks $A(m)$ and $B(m)$ of matrices A and B corresponding to the strongly connected component V^m. These matrices have the following block form:

$$A(m) = \begin{bmatrix} A^{m-1,m-1} & A^{m-1,m} \\ A^{m,m-1} & A^{m,m} \end{bmatrix}, \; B(m) = \begin{bmatrix} B^{m-1,m-1} & B^{m-1,m} \\ B^{m,m-1} & B^{m,m} \end{bmatrix} \tag{21}$$

where $A^{m-1,m}$ and $B^{m,m-1}$ are null matrices.

Proposition 5. *Let $z(1)$ and $w(1)$ be the normalized Bonacich measures of the strongly connected network with nodes set V^1. Bonacich measures $z(m)$ and $w(m)$ of the strongly connected network with node set V^m satisfy the* *following recursions, for* $1 < m \le k$*:*

$$z(m) = \frac{\beta_m A^{m,m-1}\vec{w}(m-1) + A^{m,m}\big(B^{m-1,m}\big)^T \vec{z}(m-1)}{\alpha_m\beta_m - \alpha'_m\beta'_m} \tag{22}$$

$$w(m) = \frac{\alpha_m\big(B^{m-1,m}\big)^T \vec{z}(m-1) + \big(B^{m,m}\big)^T A^{m,m-1}\vec{w}(m-1)}{\alpha_m\beta_m - \alpha'_m\beta'_m} \tag{23}$$

The Bonacich measures of the connected network with the node set $V = V^1 \cup \cdots \cup V^k$ *are given by* $\vec{z}(k) = z(1)^\frown \cdots ^\frown z(k)$ *and* $\vec{w}(k) = w(1)^\frown \cdots ^\frown w(k)$*.*

Proof. Apply the proof of Lemma 1. □

The eigenvalues α_m, β_m and constants α'_m, β'_m are different at different steps of recursion, and therefore these variables have subscripts corresponding to the recursion step.

3.2. Bonacich measures as subgame perfect equilibrium paths

Let G be a directed and connected bipartite network that has an ordered collection $\mathcal{C} = \{V^1, \ldots, V^k\}$ of strongly connected components $V^m \subset V$, $m = 1, \ldots, k$. We assume that each V^m contains at least two nodes. Consider the following extensive form game Γ associated with G.

The player set is $N = \{1_1, 1_2, \ldots, k_1, k_2\}$. The game has k stages. At stage m players m_1 (row player) and m_2 (column player) play a two-person game by choosing simultaneously nonnegative vectors z_m and w_m. First players 1_1 and 1_2 choose their strategies. Players m_1 and m_2 observe all the choices made by players i_1, i_2, for $i < m$, and then they choose their strategies z_m and w_m simultaneously.

Vector z_m (w_m) has the same number of coordinates as the matrix $A^{m,m}$ and $B^{m,m}$ in equation (21) has rows (columns). Strategies z_1 and w_1 must satisfy the norm bounds $\|z_1\| \le 1$ and $\|w_1\| \le 1$. Strategies z_m and w_m of players m_1 and m_2 must satisfy the following norm bounds for $m > 1$:

$$\|z_m\| \leq \left\| \frac{\beta A^{m,m-1}\vec{w}_{m-1} + A^{m,m}\big(B^{m-1,m}\big)^T \vec{z}_{m-1}}{\alpha\beta - \alpha'\beta'} \right\| \tag{24}$$

$$\|w_m\| \leq \left\| \frac{\alpha\big(B^{m-1,m}\big)^T \vec{z}_{m-1} + \big(B^{m,m}\big)^T A^{m,m-1}\vec{w}_{m-1}}{\alpha\beta - \alpha'\beta'} \right\| \tag{25}$$

The strategy sets of players at stage m depend on the choices made at earlier stages in the game. If constraints (24) and (25) cannot be satisfied at some stage then we interpret that the game has ended after the last stage t such that the strategy sets were still nonempty.

The payoff functions u_1^m and u_2^m of players m_1 and m_2 are given by

$$u_1^m(\vec{z}_k, \vec{w}_k) = \vec{z}_m^T A(m)\vec{w}_m, \text{ and } u_2^m(\vec{z}_k, \vec{w}_k) = \vec{z}_m^T B(m)\vec{w}_m, \tag{26}$$

where matrices $A(m)$ and $B(m)$ are given by equation (21) for $m > 1$, and $A(1) = A^{1,1}, B(1) = B^{1,1}$ in equation (21).

The Bonacich measures $z(1)^\frown \cdots {}^\frown z(t+1)$ and $w(1)^\frown \cdots {}^\frown w(t+1)$ of the connected network with node set $V^1 \cup \cdots \cup V^{t+1}$ is then found by the same way as the above.

Proposition 6. *Extensive game Γ has a unique subgame perfect equilibrium path such that the equilibrium strategies of players 1_1 and 1_2 are nonzero vectors. The equilibrium path is $\big((z(1), w(1)), \ldots, (z(k), w(k)\big)$ given by Proposition 5.*

Proof. Take any subgame perfect equilibrium such that the equilibrium strategies of players 1_1 and 1_2 are nonzero vectors. Then in every subgame perfect equilibrium the actions chosen at stage 1 are equal to the Bonacich measures $z(1)$ and $w(1)$ by Proposition 1 since the payoffs of players 1_1 and 1_2 do not depend on actions that will be chosen at stages $m > 1$.

At stage $m > 1$ the payoffs of players do not depend on actions that will be chosen at stages $t > m$. Consider the row player m_1, assume that at stages $t < m$ actions $z(t)$ and $w(t)$ given by Proposition (1). Suppose that player m_1 expects that player m_2 chooses $w(m)$. By equation (26) the payoff from action z_m to player m_1 is

$$K + z_m^T A^{m,m-1} \vec{w}(m-1) + z_m^T A^{m,m} w(m) = K + \beta z_m^T z(m), \qquad (27)$$

where $\beta > 0$. In this equation we have replaced the matrix $A(2)$ in equation (12) by the matrix $A(m)$ in equation (21). The constant K is given by $K = \vec{z}(m-1)A^{m-1,m-1}\vec{w}(m-1)$. Feasible actions z_m for player m_1 must satisfy $\|z_m\| \leq \|z(m)\|$. Hence the best reply is $z(m)$. □

References

Ballester, C., Calvó-Armengol, A. and Zenou, Y. (2006) Who's who in networks: wanted the key player. *Econometrica* **74**: 1403–1417.

Binmore, K. (1987) Nash bargaining theory I–III. In Binmore, K. and Dasgupta, P. (eds.) "The economics of bargaining'. Basil Blackwell, Oxford.

Bonacich, P. (1972) Factoring and weighting approaches to clique identification. *Journal of Mathematical Sociology* **2**: 113–120.

Bonacich, P. (1987) Power and centrality: A family of measures. *American Journal of Sociology* **92**: 1170–1182.

Bonacich, P. (1991) Simultaneous group and individual centralities. *Social Networks* **13**: 155–168.

Bonacich, P. and Lloyd, p. (2001) Eigenvector-like measures of centrality for asymmetric relations. *Social Networks* **23**: 191–201.

Brin, S. and Page, L. (1998) The anatomy of a large-scale hypertextual web search engine. *Computer Networks and ISDN Systems* **30**: 107–117.

Cabrales, A., Calvó-Armengol, A. and Zenou, Y. (2011) Social interactions and spillovers. *Games and Economic Behavior* **72**: 339–360.

Freeman, L. (1979) Centrality in social networks: Conceptual clarification. *Social Networks* **1**: 215–239.

Jackson, M.O. and Zenou, Y. (2014) Games on networks. Forthcoming in *Handbook of Game Theory Vol 4*, Young, P. and Zamir, S. (eds.), Elsevier Science, Amsterdam.

Katz, L. (1953) A new status index derived from sociometric analysis. *Psychometrika* **18**: 39–43.

Kitti, M. (2012) Axioms for centrality scoring with principal eigenvectors. ACE Discussion Paper 79.

Kleinberg, J. M. (1999) Authoritative sources in a hyperlinked environment. *Journal of the ACM* **46**: 604–632.

Nash, J.F. (1950) The bargaining problem. *Econometrica* **18**: 155–162.

Nash, J.F. (1950) Two-person cooperative games. *Econometrica* **21**: 128–140.

Rubinstein, A. (1982) Perfect equilibrium in a bargaining model. *Econometrica* **50**: 97–109.

Salonen, H (2014) Equilibria and centrality in link formation games. ACE Discussion Paper 90.

Shapley, L.S. (1953) A value for n -person games. *Annals of Mathematic Studies* **28**: 307–317.

The **Aboa Centre for Economics (ACE)** is a joint initiative of the economics departments of the Turku School of Economics at the University of Turku and the School of Business and Economics at Åbo Akademi University. ACE was founded in 1998. The aim of the Centre is to coordinate research and education related to economics.

Contact information: Aboa Centre for Economics, Department of Economics, Rehtorinpellonkatu 3, FI-20500 Turku, Finland.

www.ace-economics.fi

ISSN 1796-3133